E-mini S&P 500 Futures Contract (ES @ CME):

- Trading time: 23.25 hours/day (active trading: 6.75 hours)
- Contract size: $50 x E-mini S&P 500 futures price
- Tick size: 0.25 index points=$12.50
- **Average daily volume in 2010: 2,194,975** (for comparison: average daily volume of regular S&P 500 futures: 345,483)
Modeling coarse-grained price dynamics

Null hypothesis: **Random Walk** (Bachelier, 1900).

Stylized facts of real price time series:

- Absence of returns’ autocorrelations
- Aggregational Gaussianity
- Fat tails of distributions
- Long memory in volatility
- Intermittency and Volatility clustering
- Multifractal scaling
- Time reversal asymmetry and Leverage effect
- Gain/Loss asymmetry
- Asymmetry in time scales
- Volume-Volatility correlation
- Bubbles and crashes
Modeling HF price dynamics

Null hypothesis: **Poisson process.**

Poisson process is a point process for which number of events in a given time interval $T$ is independent from events outside the interval and described with Poisson distribution:

$$P\left[N(t + T) - N(t) = k\right] = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

where $\lambda = \text{const}$ is an *intensity* of the process.

Poisson process is characterized by an exponential distribution of inter-event times $\tau$. 
"Stylized facts" of real order arrivals

Clustering of order arrivals
Long memory in inter-trade intervals

\[ C(n) \sim n^{2\alpha-2} \]
\[ \alpha_z=0.94 \]
\[ \alpha_x=0.65 \]
\[ \alpha=0.50 \]

Multifractal scaling of inter-trade intervals

Long memory in the signs of orders

Slower-than-exponential decay of the distribution of inter-trade intervals

Modeling HF price dynamics

Extended models:

- **Clustered point processes**
  Poisson process supplemented with artificial clusters around immigrants.

- **Autoregressive Conditional Durations (ACD)**
  GARCH-type model for the inter-trade intervals:
  \[
  \tau_k = \theta_k z_k, \quad z_k > 0, \quad \mathbb{E}[z_k] = 1
  \]
  \[
  \theta_k = \alpha_0 + \sum_{i=1}^{q} \alpha_i \tau_{k-i} + \sum_{i=1}^{p} \beta_i \theta_{k-i}
  \]

- **Self-excited point processes:**
  - Linear: **Hawkes process**
  - Nonlinear, e.g. **Multifractal Stress Activation (MSA)**

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- D. Sornette, G. Ouillon (2005) PRL 94(3): 038501
Self-excited Hawkes process

Self-excited Hawkes process is the point process whose intensity \( \lambda_i(t) \) is conditional on its history:

\[
\lambda(t) = \mu + \sum_{t_i < t} \varphi(t - t_i)
\]

- **Background intensity**
- **Self-excitation part**

Traditionally the exponential memory kernel is used:

\[
\varphi(t) = \alpha e^{-\beta t}
\]

Sample realization of the Hawkes process with \( \mu=1.5, \alpha=0.4, \beta=2 \)
The branching process

Crucial parameter of the branching process is the “branching ratio” \( (n) \) - average number of “daughters” per one “mother”.

For \( n < 1 \) system is subcritical (stationary evolution)
For \( n = 1 \) system is critical (tipping point)
For \( n > 1 \) system is supercritical (with prob.>0 will explode to infinity)

The branching ratio \( (n) \) is equal to the fraction of endogenously generated events among the whole population.

\[ n = 0.88 \]
Estimation of the branching ratio

For the Hawkes process the branching ratio is given by expression:

\[ n = \int_0^\infty \varphi(t) \, dt = \frac{\alpha}{\beta} \]

Parameters of the Hawkes model could be estimated using the Maximum Likelihood method:

\[ \log L(\theta|t_1, \ldots, t_N) = -\int_0^T \lambda_t(t) \, dt + \sum_{i=1}^N \log \lambda_t(t_i) \]

which doesn’t have closed-form solution, but could be solved using numerical optimization method, such as Nelder-Mead or similar.

The quality-of-fit test is straightforward using the residual process:

\[ \tau_k = \int_0^{t_k} \lambda_t(t) \, dt \]

which should be Poisson with unit intensity if data follows the model.
Intraday seasonality

Average number of mid-price changes in 10 minutes interval during Regular Trading Hours in April 2009

Intraday profiles of estimated background activity and branching ratio in April 2009

Estimation of short-term reflexivity

Trading activity
proxied by volume and
number of mid-price changes

Dynamics of price and volatility

Rate of exogenous events (triggered
by idiosyncratic “news”)

Branching ratio that quantifies
reflexivity of the system
(fraction of endogenous events
in the system)

Flash-crash event

A Flash in The Market

Stock markets plunged suddenly on May 6 of this year and gained speed as computer programs prevented losses. But almost as quickly, the market recovered much of the decline.

1. 1:00 Volatility in some stocks increases in a down market.
2. 2:30 Unusually nervous trading pushes overall volatility up sharply; the Dow is down 2.5 percent.
3. 2:32 A program to sell $4.1 billion in E-Mini futures starts; other traders react by starting to sell.
4. 2:41 Selling in the futures market spreads to stocks; automated trading programs react to the sharp drops by shutting down.
5. 2:46 After trading in E-Mini futures is paused for five seconds, alleviating the pressure to sell, the market begins to recover.

Sources: Bloomberg (Dow industrials); Securities and Exchange Commission


April 27, 2010:
Significant fall of most of US markets following the cut of the credit rating of Greece and Portugal

May 6, 2010 ("flash-crash"):
The activity of high-frequency traders of the S&P 500 E-mini futures contracts leaded to a dramatic fall in other markets

Volume and Trading activity behave similar in both cases

Branching ratio (degree of reflexivity) reveals fundamental difference between two shocks

Exogenous vs endogenous shocks in HF

Final remarks

- In contrast to “neo-classical” theories, feedback mechanisms (reflexivity) play exceptionally important role in price dynamics.

- News plays a minor role in market volatility; most of price changes are result of internal feedback mechanisms. Due to the development of AT (and in particular HFT) endogeneity of price movements increased dramatically.

- The estimation of the branching ratio provides a novel powerful metric of endogeneity, which is much richer than standard direct measures of activity such as volume and trading rates.

- This measure allows real-time diagnostics of the market and distinguishing of exogenous (triggered by news) and endogenous (self-excited) events.