ASYMPTOTIC BEHAVIOR OF FINITE TIME RUIN PROBABILITIES OF A DEPENDENT RISK MODEL WITH CONSTANT FORCE OF INTEREST

K.K. THAMPI

SNMC-MAHATMA GANDHI UNIVERSITY
INDIA

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- \( m(t) = E\{N(t)\} \)
Cramer-Lundberg Model

- Claim sizes \( \{X_k\}_{k \in \mathbb{N}} \) are positive i.i.d. with distribution function \( F \) with \( \mu = E(X_1) \)

- \( \{N(t)\}_{t \geq 0} \) is a homogeneous PP with intensity \( \lambda > 0 \), the inter-arrival times \( \nu_1, \nu_2, \ldots \) are exponential.

- \( \{N(t)\}_{t \geq 0} \) and \( \{X_k\}_{k \in \mathbb{N}} \) are assumed to be independent processes.

- Safety loading \( \rho = c\lambda^{-1} - E(X_1) > 0 \), \( c \) - premium constant.

- \( U(t) = x + ct - S(t), \quad t \geq 0 \)

- First introduced by Lundberg (1903), complemented by Cramer (1969)
Sparre Andersen Model

- inter-arrival times \( v_1, v_2, \ldots \) form a general distribution with distribution function \( G \).

- The resulting model is a Renewal Risk model, introduced by Sparre Andersen (1957).

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**Time of ruin**

\[ \tau(x) = \inf \{ t > 0 : U(t) = x + ct - \sum_{k=1}^{N(t)} X_k < 0 \}, \quad x_k \geq 0 \]
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### Probability of ultimate ruin

$\psi(u) = \Pr\{U(t) < 0, \text{ for some } t > 0\}$
Risk Model with Interest Force

Introduced by Sundt and Tugels (1995)

- Company receives interest on it reserves.
- Assume a constant force interest $\delta > 0$.
- Surplus process under interest force.

Surplus process under interest force

$$U_{\delta}(t) = xe^{\delta t} + \int_0^t e^{\delta(t-s)} C(ds) - \sum_{n=1}^{N(t)} X_n e^{\delta(t-\tau_n)} \quad t \geq 0.$$
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### Time of ruin

\[
\tau_\delta(x) = \inf \{ t : U_\delta(t) < 0 \}
\]

### Finite time ruin Probability

\[
\psi_\delta(x, T) = \Pr \{ \tau_\delta(x) < T \mid U_\delta(0) = u \}
\]
Real Situation?

Why focussing on classical model?

- Mathematical tractability
- Simplicity of model and assumptions

Questionable?

Independence assumptions

Inter-claim distribution.

Result?

Search for more general models

Still use independence assumptions.

Why?

Available methods do not work without assumptions.

What is the Real situation?

Most of the claims are dependent.

Claims sizes and inter-claim times are also dependent.

Frequency of large claims is less and inter-claim times tend to longer.

Return rates for investments are also dependent.
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Relaxing Assumptions?

Relax

- Independence assumption.
- Claim sizes and return rates of interest are heavy tailed.

**Example:** Catastrophe insurance, Financial and environmental data.
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Rethink?

Modeling of financial concept was based on the assumption that financial data was distributed normally. It was Mandelbrot (1963) who found that the financial data has high degree of skewness and huge tails which was not the characteristic of Gaussian distribution. The distribution with huge tails was termed as ”Heavy tailed distributions”, which can accommodate large claims (extreme values)
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- Heavy tailed claim sizes make large fluctuations in the risk process
- Heavy tail concept has the right level of generality of risk measurement in insurance and financial models.
Heavy tailed Distributions

Role?
- Modelling of insurance and financial Risk processes.
- Epidemiological spread.
- Computer and communication networks, Signal processing .....etc.

Definition
A distribution $F$ on $\mathbb{R}$ is said to be heavy tailed if
$$\int_{-\infty}^{\infty} e^{tx} F(dx) = \infty,$$
for all $t > 0$.

$F$ is heavy tailed iff its tail function $F$ fails to be bounded by any exponentially decreasing function (Sergy Foss et.al (2011)).

There are other definitions for heavy tails in literature (Resnick (2007)).

Tail Function
$F(x) = F(x, \infty)$, where $F$, the distribution function.

$F$ has right support if $F(x) > 0$ for all $x$.
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RUIN PROBABILITIES OF A DEPENDENT RISK MODEL
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$\overline{F}(x) = F(x, \infty)$, where $F$, the distribution function. $F$ has right support if $\overline{F}(x) > 0$ for all $x$. 
Subexponential Distribution

A special class of heavy tailed distribution denoted by $S$

1. $\Pr\{X_1 + X_2 + \ldots + X_n > x\} \geq \Pr\{\max(X_1, X_2, \ldots, X_n) > x\} = 1 - F_n(x) \sim nF(x)$

- Importance?

- Tail probabilities of the sum and the maximum of one of the individual random variables \{X_1, X_2, \ldots, X_n\} are asymptotically of the same order known as the Big Jump.

- The subexponential claims can account for large fluctuations of the surplus process of an insurance company.

- This important feature explains the relevance of subexponential distributions in the modelling heavy tailed phenomena in insurance and finance.
A special class of heavy tailed distribution denoted by $S$

$X_1, X_2, \ldots, X_n$ be independent random variable with distribution $F$. Then

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$$= 1 - F^{*n}(x)$$

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$\sim$ means the quotient of LHS and RHS tend to 1 as $x \to \infty$
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SubClasses

Well known Subexponential distributions
- Lognormal
- Pareto
- Burr
- Benktander type I and type II
- Weibull (shape parameter $\alpha < 1$)
- Loggamma
- Cauchy
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Different subclasses of heavy tailed

- Long tailed $L \iff \lim_{x \to \infty} \frac{F(x+y)}{F(x)} = 1$, for $y > 0$
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- Dominated varying class $D \implies \lim_{x \to \infty} \frac{F(xy)}{F(x)} < \infty$ for $0 < y < 1$. 
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- Consistently varying tails $C \implies \lim_{y \downarrow 1} \lim_{x \to \infty} \inf \frac{F(xy)}{F(x)} = 1$
- Extended Regularly Varying tails $ERV \implies$
  $$y^{-\beta} \leq \lim_{x \to \infty} \inf \frac{F(xy)}{F(x)} \leq \lim_{x \to \infty} \sup \frac{F(xy)}{F(x)} \leq y^{-\alpha}, \ y \geq 1$$
An important subclass (Discussed in this paper) of heavy tailed, denoted by \( \mathcal{R} \).

**Definition**

A subexponential distribution \( F \), denoted by \( F \in \mathcal{R} \), if for some \( \alpha > 0 \),

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\lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^{-\alpha}, \text{ for } y > 0.
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\( F \in R_{-\alpha}, \quad R = \bigcup_{\alpha \geq 0} R_{-\alpha} \)
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\( F \in \mathcal{R}^{-\alpha}, \mathcal{R} = \bigcup_{\alpha \geq 0} \mathcal{R}^{-\alpha} \)

The following inclusion relationship is valid.

- \( \mathcal{R} \subset \mathcal{ERV} \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{S} \subset \mathcal{L} \)
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The following inclusion relationship is valid.

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**Definition**

**Uniform Convergence**

If $F \in \mathcal{R}_{\alpha}$, then for $0 < a \leq b < \infty$
\[
\lim_{x \to \infty} \frac{F(xy)}{F(x)} = y^{-\alpha}, \text{ uniformly on } y \in [a, \infty).
\]
The random variables $X_1$ and $X_2$ are said to be WND if there exist a $C > 1$ such that $f(x_1, x_2) \leq Cf_1(x_1)f_2(x_2)$.
Weakly Negatively Dependence (WND)

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**Lemma**

*Let $X_1$ and $X_2$ be two WND random variables, then for all real numbers $x_1, x_2$, we have*

\[
\Pr\{X_1 \leq x_1, X_2 \leq x_2\} \leq C \Pr\{X_1 \leq x_1\} \Pr\{X_2 \leq x_2\}
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*Equivalently*

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\Pr\{X_1 > x_1, X_2 > x_2\} \leq C \Pr\{X_1 > x_1\} \Pr\{X_2 > x_2\}
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**Pairwise WND**

The sequence of random variables of random variables $\{X_1, X_2, \ldots\}$ is pairwise WND if for any positive integer $i \neq j$, the random variables $X_i$ and $X_j$ are WND.
Examples of WND & an important Lemma

- The random variables $X_1$ and $X_2$ have half normal distribution with
  
  \[ f_{X_1,X_2}(x_1, x_2) = \frac{1}{\pi \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2) \right\} \]
  
  can be converted to WND using a simple transformation.

- The so called Farlie-Gumbel-Morgenstern (FGM) family of distribution provides a simple mechanism to construct practically interesting pair of random variables that are WND.
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- The so called Farlie-Gumbel-Morgenstern (FGM) family of distribution provides a simple mechanism to construct practically interesting pair of random variables that are WND.

Lemma

Let $\{X_k, k \geq 1\}$ be a sequence of non-negative WND random variables with common distribution function $F$. If $F \in \mathcal{R}$, then the relation

$$\Pr\left\{\sum_{k=1}^{n} c_k X_k > x\right\} \sim \sum_{k=1}^{n} \Pr\{c_k X_k > x\}$$

holds uniformly for $c_n := (c_1, c_2, \ldots, c_n)$
Asymptotic behavior of ruin probability has been extensively studied in literature. Under the condition \( \{X_k\}_{k \geq 1} \) are i.i.d with common distribution \( F \in \mathcal{R}_{-\alpha} \) for some \( \alpha > 0 \) and \( \{T_k\}_{k \geq 1} \) are also i.i.d.
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Kluppelburg and Stadtmüller (1998) obtained asymptotic behavior of infinite time probability under independence assumptions.

Cheng and Tang (2007) obtained asymptotic formula for finite time ruin probability for pairwise NQD (negatively quadrant dependent) claims. Papers are in recent literature also.
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Assmussen ; Kalashinikov, Konstantinides (2000), Tang (2004) have studied the asymptotic behavior of ruin probability to larger classes.
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Tang (2005) obtained finite time ruin probability of the compound Poisson model with constant interest force.
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Papers are in recent literature also.
It is the aim of this paper to study a non-standard compound renewal risk model which extends the classical renewal model in simple, but more realistic way. In the usual renewal risk model, only one claim is assumed at a claim point. But in this paper we assume that there are several WND claims at a claim point $\tau_j$. It is quite natural that the individual caused by same accident are dependent, but claims caused by different accidents are mutually independent.
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**Assumption (A1)**

- The claim sizes $\{X_k, \ k \geq 1\}$ form a sequence of non-negative and WND random variables with distribution
  
  \[ F(x) = \Pr\{X \leq x\} = 1 - \overline{F}(x) \]  
  
  with  
  \[ E\{X_1\} = \mu < \infty \]
Model

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**Assumption A(2)**

- The interarrival times $\{\nu_k, \ k \geq 1\}$ form another sequence of WND and non-negative random variables which is essentially independent of the random variables $\{X_k, \ k \geq 1\}$. 
Random Sum

Assumption (A3)
The individual claim sizes and the claim number caused by the \(n\)th accident at time \(t_n\) are \(\{X_{nk}, k \geq 1\}\) and \(K_n\) respectively. Here \(\{X_{nk}, k \geq 1\}\) are non-negative WND random variables and \(K_n\) are non-negative integer valued WND random variables with common distribution \(H\). Assume that \(\{\nu_n, n \geq 1\}\), \(\{K_n, n \geq 1\}\) and \(\{X_{nk}, k \geq 1\}\) are mutually independent.

The total claim amount at the time \(t_n\) is \(S_{nk} = \sum_{j=1}^{k} X_{nj}, 0 \leq k \leq K_n\).

We write \(K, X\) etc for generic random variable with same distribution as \(K_n, X_{nk}\) etc.
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We can also write

\[ S_0 = 0, \quad S_n = \sum_{k=1}^{n} X_k \]

where \( \{X_n, n \geq 1\} \) is WND sequence with same distribution as \( \{X_{nj}, j \geq 1\} \). The study of the tail behavior of the random sum \( \{S_n, n \geq 1\} \) is of fundamental interest of this paper.

This setting is most relevant in insurance portfolio, for instance, to earthquake insurance or tsunami insurance, featuring potentially a large number of bounded claim sizes. The tail of the total claim amount is mainly dependent on the tail of the claim size (Robert & Segers (2008)).
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Now the total claim amount upto time \( t \geq 0 \) is

\[ \sum_{n=1}^{N(t)} S_{nk} = \sum_{n=1}^{N(t)} \sum_{j=1}^{k} X_{nj} \]
Lemma

Let \( \{X_k, \ k \geq 1\} \) be a sequence WND and regularly varying random variables with index \( \alpha > 0 \) and \( N \) be an integer valued random variable with tail distribution \( T(x) = \text{Pr}\{N > x\} \) and \( E(N) < \infty \) which is independent of \( \{X_k, \ k \geq 1\} \). Let \( \mu > 0 \) and one of the following two conditions be satisfied.

(1) \( \text{Pr}\{N > x\} = o(F(x)) \)
(2) \( T \in \mathcal{R} \)

In the former case

\[
\text{Pr}\{S_N \geq x\} \sim E(N)F(x)
\]

In the latter case

\[
\text{Pr}\{S_N \geq x\} \sim E(N)\overline{F}(x) + \text{Pr}\{N > x\mu^{-1}\}
\]
\[ \Pr\{N > x\} = o(\Pr\{X > x\} \text{ as } x \to \infty \text{ to mean that} \]

\[ \lim_{x \to \infty} \frac{\Pr\{N > x\}}{\Pr\{X > x\}} = 0 \]

Intuitively, this means that \( X \) has heavier tail than \( N \).

The tail of \( S_N \) has been studied for the class of subexponential distribution of \( X \) under the assumption that \( \Pr\{N > x\} \) decays exponentially fast, then

\[ \Pr\{S_N \geq x\} \sim E(N)\overline{F}(x) \]

see Embrechts et.al(1997).

In the lemma stated above, the same condition is valid under weak conditions on the tail of \( N \).

To prove this lemma, we need the asymptotic properties of Regularly varying tails and their functions.
Asymptotic Properties of Function of Distributions

Let $F$ be the distribution of a random variable $X$ with $F \in \mathcal{R}$. $\overline{F}(x)$ denote the tail of this distribution: $\overline{F}(x) = F(x, \infty)$. Let $g(s) = E\{e^{isX}\}$ be the characteristic function and let $A(\omega)$ be a function of the complex variable $\omega$.

**Lemma**

Let the distribution $F$ of the random variable $X$ be subexponential and let $A(\omega)$ be analytic in the disk $|\omega| \leq 1$. There exist a finite measure $\mathbf{A}$ such that

$$A(g(s)) = \int e^{isx} \mathbf{A}(dx)$$

$$\overline{A}(t) = A(t, \infty) \sim A'(1)\overline{F}(t)$$

**Proof** (Refer Borovkov(2008))
We assume that

\[ A(\omega) = \sum_{k=0}^{\infty} a_k \omega^k \]

where \( \sum_{k=0}^{\infty} a_k \) is absolutely convergent. Given that \( a_k \geq 0 \),

\[ A(1) = \sum_{k=0}^{\infty} a_k < \infty \quad A'(1) = \sum_{k=0}^{\infty} k a_k > 0 \]

Without loss of generality, we assume that \( A(1) = 1 \) and consider \( A(g(s)) \) as a characteristic function of a random variable \( s \) for which

\[ E\{e^{itS}\} = A(g(t)) \]

where

\[ S = S_N = \sum_{k=1}^{N} X_k \]

where the random variable \( N \) is independent of \( \{X_k, k \geq 1\} \).

\[ \Pr\{N = k\} = a_k \]. Therefore \( A'(1) = E\{N\} \)
Proof of the Lemma

Consider

\[ A(x) = \sum_{n=0}^{\infty} a_n \Pr\{S_n \geq x\} = \Sigma_1 + \Sigma_2 + \Sigma_3 \]  (1)

\[ \Sigma_1 = \sum_{n \leq n_1}, \quad \Sigma_2 = \sum_{n = n_1 + 1}^{(1-\epsilon)\mu^{-1}x}, \quad \Sigma_3 = \sum_{n > (1-\epsilon)\mu^{-1}x} \]  (2)

\[ \Sigma_1 \sim A'(1)F(x) \sim E(N)F(x) \]

For \( \Sigma_2 \)

Assume \( \Pr\{N > x\} = o(\Pr\{X > x\}) \) is met.

\[ \frac{\Pr\{X > x - n\mu\}}{\Pr\{X > x\}} < c \quad \text{for} \quad n \leq (1-\epsilon)\mu^{-1}x \]  (3)

\[ \frac{\Pr\{X > x - n\mu\}}{\Pr\{X > x\}} \to 1 \quad \text{for} \quad n = o(x) \]  (4)
\[ A(x) = A'(1)\bar{F}(x)(1 + o(1)) + \Sigma_3 \]

where \( |\Sigma_3| \leq \Pr\{N > (1 - \epsilon)\mu^{-1}x\} \sim c_1 \Pr\{N > x\} = o(\bar{F}(x)). \)

(This proof uses Large Deviation results.)
\[ A(x) = A'(1)\bar{F}(x)(1 + o(1)) + \Sigma_3 \]

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(This proof uses Large Deviation results.)

For the latter part:

Let \( T \in \mathcal{R} \), by LLN \( \frac{S_n}{n} \to \mu \) as \( n \to \infty \). We have \(\Pr\{S_n \geq x\} \to 1\) for \( n \geq x\mu^{-1}(1 + \epsilon)\). Assuming for definiteness that \( T(x\mu^{-1}) > 0 \)

\[ T(x\mu^{-1}(1 + \epsilon))(1 + o(1)) \leq \Sigma_2 \leq T(x\mu^{-1}(1 - \epsilon)) \]

\( \epsilon \) arbitrary, \( \Sigma_2 \sim T(x\mu^{-1})\). Proved!
Consider a WND non-standard compound renewal risk model introduced above. Assume that $A_1, A_2,$ and $A_3$ are met. Let $\{X_k, k \geq 1\}$ be WND with common distribution $F \in \mathcal{R}$ for some $\alpha > 0$. If $EN < \infty$ and $\Pr\{N > x\} = o(F(x))$ Then

$$\psi_\delta(x, T) \sim \int_0^T ENF(xe^{\delta t})dm(t)$$

If $H \in \mathcal{R}$, then

$$\psi_\delta(x, T) \sim \int_0^T ENF(xe^{\delta t})dm(t) + \int_0^T H(\mu^{-1}xe^{\delta t})dm(t)$$

**Proof:** When the assumptions $A_1, A_2$ and $A_3$ are true, then the discounted surplus process of the insurance company can be expressed in the form

$$U_\delta(t) = x + \int_0^T e^{-\delta s} C(ds) - \sum_{n=1}^{N(t)} S_{nk} e^{-\delta \tau_n}$$
By the lemma stated above, it is easy to prove that if \( \Pr\{N > x\} = o(F(x)) \), then

\[
\psi_\delta(x, t) \sim \int_0^T E\bar{N}\bar{F}(xe^{\delta t})dm(t)
\]

if \( H \in \mathcal{R} \)

\[
\psi_\delta(x, T) \sim \int_0^T (E\bar{N}\bar{F}(xe^{\delta t}) + \bar{H}(x\mu^{-1}e^{\delta t}))dm(t)
\]

Remark

Assume that \( A_1 \) and \( A_2 \) are satisfied and \( \{X_k, k \geq 1\} \) and \( \{\nu_k, k \geq 1\} \) are mutually independent random variables. If \( \Pr\{N > x\} = o(F(x)) \), then

\[
\psi_\delta(x, T) \sim \int_0^T F(xe^{\delta t})dm(t)
\]

If \( H \in \mathcal{R} \), then

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\psi_\delta(x, T) \sim \int_0^T (F(xe^{\delta t}) + \bar{H}(x\mu^{-1}e^{\delta t}))dm(t)
\]
The classical results for Large Deviation states that $F \in R(\alpha)$

$$\Pr\{S_n - E(S_n) > x\} \sim \Pr\{\max(X_1, X_2, \ldots, X_n) > x\}$$

$$\sim nF(x), \quad n \to \infty$$

which relation holds uniformly for $x > \gamma n$ for any fixed $\gamma > 0$.

$$\sup_{x \in (\gamma n, \infty)} \left| \frac{\Pr(S_n - E(S_n))}{nF(x)} - 1 \right| \to 0, \quad n \to \infty$$
MY SINCERE THANKS TO

- ALL MEMBERS OF THE ORGANIZING COMMITTEE
- ALL PARTICIPANTS OF THIS CONFERENCE

BEST OF LUCK TO ALL