Risk, Return, and Ross Recovery

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The risk return relation is a staple of modern finance.

When risk is measured by volatility, it is well known that option prices convey risk.

Yet when it comes to predicting the average return, the conventional wisdom is that option prices are silent in this respect.

In 2011 Steve Ross began circulating a working paper which challenges this conventional wisdom.

The Ross Recovery theorem says that under some conditions, option prices can be used to obtain expected return, and a whole lot more.
Consider a single period model with a finite number of discrete states.

Arrow Debreu securities are fictitious derivative securities that pay one dollar if a given state occurs and zero otherwise.

Risk-neutral probabilities are just forward prices of these Arrow Debreu securities.

Under no arbitrage, these forward prices are positive and sum to one.

Risk neutral probabilities are affected by both the likelihood that the state occurs and the aversion that investors have to the state.

If you know only the risk-neutral probabilities, it is impossible to separate beliefs from risk-aversion.
Suppose that states are just the future price of some optioned asset e.g. S&P 500.

Breeden and Litzenberger (1978) showed that if we know the forward prices of European options struck at all possible levels for its underlying, then their second strike difference gives us the risk-neutral probabilities of the underlying at expiration.

The idea also works when the underlying asset price can take values in a continuum. The second strike difference just becomes a second strike derivative.

These results are attractive because they are model-free.
If one is willing to work with a model, even more can be said.

When a Markovian intertemporal model, e.g. Black Scholes, CEV, or CGMY, is fit to option prices, we also get risk-neutral transition probabilities.

Hence, not only would we know the price of an Arrow Debreu security that pays off if say S&P500 stays between 1000 and 1100 in 1 year, but we would also know the price of an Arrow Debreu security that only pays off if S&P500 furthermore stays between 1000 and 1200 in 2 years.

Thus a model can be used to convert the market prices of a path-independent payoff such as options into the model prices of path-dependent securities.

In fact, the path-dependent security described above is similar to one that actually trades called a wedding cake.
In 2011, Steve Ross began circulating a working paper called “The Recovery Theorem”

The theorem lists sufficient conditions under which a known risk-neutral transition probability matrix can be converted into a real-world transition probability matrix.

The latter matrix governs transitions of a Markovian state variable $X$ which determines aggregate consumption.

Ross uses S&P500 as a proxy for $X$ and produces a forecast from SPX option prices.
The conclusions of Ross’s model have strong implications for both financial theory and for the equity index options industry.

As the late great economist Paul A. Samuelson famously said, “The stock market has forecast nine of the last five recessions.”

It will certainly be interesting to see if the stock index options market can produce a better record than its underlying stock market.
As already indicated, the Ross Recovery theorem says that under some conditions, a set of option prices can be used to obtain the expected return of its underlying asset for all horizons out to the longest maturity.

Those of us raised on the Black Merton Scholes paradigm find Ross’s claims to be startling.

If one can value options without knowledge of expected return, then how can one use option prices to infer expected return?

On the other hand, everyone knows that the greater the risk, the greater the expected return. If expected returns are increasing in volatility, then higher option prices imply higher volatility and higher expected return.

Jiming Yu and I set out to get to the bottom of this conundrum.
Digging deeper, we learned that Ross makes some assumptions on preferences that seem hard to verify.

In particular, Ross places his structure on the preferences of a "representative agent".

According to wikipedia:

*economists sometimes say a model has a representative agent when agents differ, but act in such a way that the sum of their choices is mathematically equivalent to the decision of one individual*
Ross also places structure on something called a state variable.

According to wikipedia:

*In control systems engineering, a state variable is one of the set of variables that are used to describe the mathematical "state" of a dynamical system. Intuitively, the state of a system describes enough about the system to determine its future behaviour.*

To an economist, a state variable describes the set of states that Arrow Debreu securities index. Once one knows the realization of the state variable, there is no residual uncertainty in the entire economy.
Theorem 1 in Ross (2011) states that

- if the utility function of the representative investor is state independent and intertemporally additively separable and:
- if the state variable is a time homogeneous Markov process $X$ with a finite discrete state space, then:
- one can recover the real-world transition probability matrix of $X$ from an assumed known matrix of Arrow Debreu state prices.
Yes, But...

- We checked every line in Ross’ proof and in the end had to agree that he had proved his theorem.
- However, being stuck in the trenches doing CVA and such, we have to confess that we have no idea whether a representative agent even exists.
- If a representative agent does exist, we doubt that even he would know whether his utility function was additively separable or not. (Is yours? - be honest).
- As Bertrand Russell once said:

  Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.
To summarize, Jiming and I were intrigued by Ross’ claim that derivative security prices can be used to forecast their underlying, but doubtful whether we could:

1. be sure the representative investor’s utility function is additively separable, or
2. find a state variable rich enough to describe the world economy, or
3. find derivative securities written on that state variable.

Fortunately, there is a concept in the financial literature that lets us overcome all of these hurdles.
In 1990, John Long introduced the notion of a *numeraire portfolio*.

A numeraire is any self-financing portfolio whose price is always positive.

Long showed that if any set of assets is arbitrage-free, then there always exists a numeraire portfolio comprised of just these assets.

The defining property of this numeraire portfolio is the following surprising result:

*If the value of the numeraire portfolio is used to deflate each asset’s dollar price, then each deflated price evolves as a martingale under the real-world probability measure.*
Proof That the Numeraire Portfolio Exists

Consider an imaginary world with a default-free money market account (MMA) whose balance $S_{0t}$ grows at rate $r_t$, one or more risky assets with spot prices $S_{it}, i = 1, \ldots, n$, and no arbitrage.

These assumptions imply the existence of a martingale measure $Q$, equivalent to $P$, under which each $r-$discounted security price, $e^{-\int_0^t r_s ds} S_{it}$, evolves as a martingale, i.e.

$$E^Q \frac{S_{iT}}{S_{0T}} | \mathcal{F}_t = \frac{S_{it}}{S_{0t}}, \quad t \in [0, T], i = 0, 1, \ldots, n.$$

Let $M$ be the positive martingale used to create $Q$:

$$E^P \frac{M_T M_t}{M} \frac{S_{iT}}{S_{0T}} | \mathcal{F}_t = \frac{S_{it}}{S_{0t}}, \quad t \in [0, T], i = 0, 1, \ldots, n.$$
Proof That the Numeraire Portfolio Exists (Con’d)

- Recall from the last slide that $M$ is the positive martingale used to create $Q$:

  \[ E^{P} \frac{M_{T}}{M_{t}} \frac{S_{iT}}{S_{0T}} | \mathcal{F}_{t} = \frac{S_{it}}{S_{0t}}, \quad t \in [0, T], i = 0, 1, \ldots, n. \]

- Fact: the positive martingale $M$ used to create $Q$ from $P$ has the property that its reciprocal $\frac{1}{M}$ is a positive local martingale under $Q$. Let $L$ denote the product of the money market account $S_{0}$ and this reciprocal:

  \[ L_{t} \equiv \frac{S_{0t}}{M_{t}}, \quad \text{for } t \in [0, T]. \]

$L$ is clearly a positive stochastic process. Since $L$ is just the product of the MMA and the $Q$ martingale $\frac{1}{M}$, $L$ grows in $Q$ expectation at the risk-free rate. As result, $L$ is the value of some self-financing portfolio. Multiplying both sides of the top equation by $M_{t}$ and substituting in $L$ implies:

\[ E^{P} \frac{S_{iT}}{L_{T}} | \mathcal{F}_{t} = \frac{S_{it}}{L_{t}}, \quad t \in [0, T], i = 0, 1, \ldots, n. \]

- In words, $L$ is the value of the numeraire portfolio that Long introduced.
Long’s discovery of the numeraire portfolio allows one to bypass the rather abstract notion of an equivalent martingale measure $\mathbb{Q}$.

Even more importantly, Long showed that for any numeraire portfolio, its risk premium IS its instantaneous variance.

In contrast, for any other asset in the arbitrage-free set, the risk premium is the instantaneous covariance of the asset’s return with that of the numeraire portfolio.

It follows that if one can determine the implied instantaneous variance of the numeraire portfolio, then one can at least determine its risk premium.

If one furthermore knows the riskfree rate, one then knows the expected return on the numeraire portfolio.

If one can also determine the covariance of each asset’s return with the numeraire portfolio, then that asset’s expected return can also be determined.
Using Long’s numeraire portfolio, we replace Ross’s restrictions on the form of preferences with our restrictions on the form of beliefs.

More precisely, we suppose that the prices of some given set of assets are all driven by a univariate time-homogenous bounded diffusion process, $X$.

Letting $L$ denote the value of the numeraire portfolio for these assets, we furthermore assume that $L$ is also driven by $X$ and $t$ and that $(X, L)$ is a bivariate time homogenous diffusion.

We show that these assumptions determine the real world dynamics of all assets in the given set.

The set of assets can be a strict subset of those in the world economy. We refer to $X$ as a driver rather than a state variable to be clear on this important point.
Our Assumptions

- We assume no arbitrage for some finite set of assets which includes a money market account (MMA).

- As a result, there exists a risk-neutral measure $Q$ under which prices deflated by the MMA evolve as martingales.

- Under $Q$, the driver $X$ is a time homogeneous bounded diffusion:

$$dX_t = b(X_t)dt + a(X_t)dW_t, \quad t \in [0, T].$$

where $X_0 \in (\ell, u)$ and where $W$ is standard Brownian motion under $Q$.

- We also assume that under $Q$, the value $L$ of the numeraire portfolio solves:

$$\frac{dL_t}{L_t} = r(X_t)dt + \sigma(X_t)dW_t, \quad t \in [0, T].$$

- We know the functions $b(x)$, $a(x)$, and $r(x)$ but not $\sigma(x)$. How to find it?
Value Function of the Numeraire Portfolio

- Recalling that $X$ is our driver, we assume:

$$L_t \equiv L(X_t, t), \quad t \in [0, T],$$

where $L(x, t)$ is a positive function of $x \in \mathbb{R}$ and time $t \in [0, T]$.

- Applying Itô’s formula, the volatility of $L$ is:

$$\sigma(x) \equiv \frac{1}{L(x, t)} \frac{\partial}{\partial x} L(x, t)a(x) = a(x) \frac{\partial}{\partial x} \ln L(x, t).$$

- Dividing by $a(x) > 0$ and integrating w.r.t. $x$:

$$\ln L(x, t) = \int_{x}^{\infty} \frac{\sigma(y)}{a(y)} dy + f(t), \text{ where } f(t) \text{ is the constant of integration.}$$

- Exponentiating implies that the value of the numeraire portfolio separates multiplicatively into a positive function $\pi(\cdot)$ of the driver $X$ and a positive function $p(\cdot)$ of time $t$:

$$L(x, t) = \pi(x)p(t),$$

where: $\pi(x) = e^{\int_{x}^{\infty} \frac{\sigma(y)}{a(y)} dy}$ and $p(t) = e^{f(t)}$. 
Separation of Variables

- The numeraire portfolio value function $L(x, t)$ must solve a PDE to be self-financing:

$$
\frac{\partial}{\partial t}L(x, t) + \frac{a^2(x)}{2} \frac{\partial^2}{\partial x^2}L(x, t) + b(x) \frac{\partial}{\partial x}L(x, t) = r(x)L(x, t).
$$

- On the other hand, the last slide shows that this value separates as:

$$
L(x, t) = \pi(x)p(t).
$$

- Using Bernoulli’s classical separation of variables argument, we know that:

$$
p(t) = p(0)e^{\lambda t},
$$

and that:

$$
\frac{a^2(x)}{2} \pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda \pi(x), \quad x \in [\ell, u].
$$
Recall the ODE on the last slide:

\[
\frac{a^2(x)}{2} \pi''(x) + b(x)\pi'(x) - r(x)\pi(x) = -\lambda \pi(x), \quad x \in [\ell, u].
\]

Here \( \pi(x) \) and \( \lambda \) are unknown. This can be regarded as a regular Sturm Liouville problem.

From Sturm Liouville theory, we know that there exists a principal eigenvalue \( \rho \), smaller than all of the other eigenvalues, and an associated positive eigenfunction, \( \phi(x) \) which is unique up to positive scaling.

All of the eigenfunctions associated to the other eigenvalues switch signs at least once.

One can numerically solve for both the first eigenvalue \( \rho \), smaller than all of the others, and its associated positive eigenfunction, \( \phi(x) \). The positive eigenfunction \( \phi(x) \) is unique up to positive scaling.
Recall that $\rho$ is the known first eigenvalue and $\phi(x)$ is the first eigenfunction, positive and known up to a positive scale factor.

As a result, the value function of the numeraire portfolio is also known up to a positive scale factor:

$$L(x, t) = \phi(x)e^{\rho t}, \quad x \in [\ell, u], t \in [0, T].$$

As a result, the volatility of the numeraire portfolio is uniquely determined as:

$$\sigma(x) = a(x)\frac{\partial}{\partial x} \ln \phi(x), \quad x \in [\ell, u].$$
Long (1990) showed that the *real world* dynamics of \( L \) are given by:

\[
\frac{dL_t}{L_t} = \left[ r(X_t) + \sigma^2(X_t) \right] dt + \sigma(X_t) dB_t, \quad t \geq 0,
\]

where \( B \) is a standard Brownian motion under the real world probability measure \( \mathbb{P} \).

- In words, the risk premium of the numeraire portfolio is simply \( \sigma^2(x) \).
- Since we have determined \( \sigma(x) \) on the last slide, the risk premium of the numeraire portfolio has also uniquely determined.
- The market price of Brownian risk is simply \( \sigma(x) \), which we also know.
Real World Dynamics of the Driver

- From Girsanov’s theorem, the dynamics of the driver $X$ under the real world probability measure $\mathbb{P}$ are:

$$dX_t = [b(X_t) + \sigma(X_t)a(X_t)]dt + a(X_t)dB_t, \quad t \geq 0,$$

where recall $B$ is a standard Brownian motion under the real world probability measure $\mathbb{P}$.

- Hence, we know the real world dynamics of the driver $X$.

- We still have to determine the real world transition density of the driver $X$. 

From the change of numeraire theorem, the Radon Nikodym derivative \( \frac{dP}{dQ} \) is:

\[
\frac{dP}{dQ} = e^{-\int_0^T r(X_t) dt} \frac{L(X_T, T)}{L(X_0, 0)} = \frac{\phi(X_T)}{\phi(X_0)} e^{\rho T} e^{-\int_0^T r(X_t) dt},
\]

since \( L(x, t) = \phi(x) e^{\rho t} \).

Let \( dA \equiv e^{-\int_0^T r(X_t) dt} dQ \) denote the assumed known Arrow Debreu state pricing density:

\[
dP = \frac{\phi(X_T)}{\phi(X_0)} e^{-\rho T} e^{-\int_0^T r(X_t) dt} dQ = \frac{\phi(X_T)}{\phi(X_0)} e^{-\rho T} dA.
\]

As we know the ratio \( \frac{\phi(X_T)}{\phi(X_0)} \), the positive function \( e^{-\rho T} \), and the Arrow Debreu state pricing density \( dA \), we know \( dP \), the real-world transition PDF of \( X \).
We highlight Ross’s Theorem 1 and propose an alternative preference-free way to derive the same financial conclusion.

Our approach is based on imposing stationarity on the real world dynamics of the numeraire portfolio when it is driven by a bounded diffusion.

We showed how separation of variables allows us to separate beliefs from preferences.

Since our results apply only to bounded diffusions, they can’t be used to determine whether option prices forecast real world returns in a model like Black Scholes where diffusions are unbounded.
We suggest the following extensions for future research.

1. Extend this work to two or more driving state variables.
2. Explore whether Ross recovery is possible on unbounded domains.
3. Explore the extent to which Ross’s conclusions survive when the driving process $X$ is generalized into a semi-martingale.
4. Explore what restrictions on $\mathbb{P}$ can be obtained when markets are incomplete.
5. Implement and test.