Optimal Asset Liquidation using Limit Order Book Information

Sasha Stoikov
(joint work with Rolf Waeber)

Operations Research & Information Engineering, Cornell University

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Optimal Liquidation

How to liquidate $X$ shares of an asset?

1. **Macroscopic** time scale:
   - Horizon $\bar{T} > 0$ over which the shares $X$ need to be liquidated.
   - Depends on *long term* variables: average daily volume, strategic considerations, news events, ...

2. **Mesoscopic** time scale:
   - Trade schedule $0 \leq t_0 \leq t_1 \ldots \leq t_i \leq \ldots \leq t_n = \bar{T}$ for the “child” trades.
   - Depends on *medium term* variables: volatility of the stock, risk aversion of the trader, price impact considerations, ...

3. **Microscopic** time scale:
   - Within a time interval $(t_i, t_{i+1}]$, what is the *timing* and the *type of order* used to liquidate the “child” trade?
   - Depends on *short term* variables: limit order book information.
Literature: Mesoscopic Time Scale

1. Almgren and Chriss (1998)
   - Objective: maximize risk-adjusted revenues
     \[ E(\mathcal{R}_T^x) - \lambda V(\mathcal{R}_T^x). \]
     - \( \mathcal{R}_T^x \) is the revenue from liquidation and \( x = (x_0, x_1, \ldots, x_T) \) is a deterministic trade schedule.
     - Solution: \( x_t = \frac{\sinh(\kappa(T-t))}{\sinh(\kappa T)} X \), for \( t \in \{0, \ldots, T\} \), and \( \kappa \) depends on price volatility, risk aversion, and price impact.

2. Schied, Schoneborn and Tehranchi (2010)
   \[ \sup_{X \in \chi} E[u(\mathcal{R}_T^X)] \]
   - \( X \) is the optimal control within a class \( \chi \) of stochastic controls.
   - If the utility function is exponential, \( X \) is a deterministic function of time.
The trade schedule
Microscopic Time Scale

- We assume that the trade schedule is \textit{given}.
- The goal is then to liquidate one lot (the shares $x_t$) in the time window $(t_i, t_{i+1}]$, i.e., what is the optimal time $\tau$ in $[0, T]$ to sell the lot, where $T = t_{i+1} - t_i > 0$.
- $T$ is typically short, e.g., 1 minute.
- For such short time periods, observing the limit order book can be very advantageous in identifying good liquidation times.
- However, \textit{latency} in the trade execution can diminish this advantage!
Latency

- Latency arises in every trade execution:
  1. Time of datafeed to travel from exchange to execution machine;
  2. The algorithm making a decision;
  3. The order being sent back to the market.

- Latency has no effect on deterministic trade schedules.

- In our model the algorithm will take into account that if a market order is sent at time $t$ it will actually be executed at the best price available at time $t + l$, for latency $l > 0$.

- This worsen the performance of our optimal liquidation algorithm, thus allowing us to quantify the cost of latency.
Outline

1. Optimal liquidation:
   - The efficient price process.
   - Optimal stopping problem.
   - The trade and no-trade regions.

2. Trading with latency.

3. Dynamic programming approximation.

4. Backtesting strategy on TAQ data.

5. Conclusions and future research.
The Efficient Price Process

- The **efficient** or “true” price process

\[ S(t) = S^b(t) + \theta(t), \]

where \( S^b(t) \) is the bid price and \( \theta(t) \) is the **imbalance process**:

\[ \theta(t) = g \left( \frac{B(t)}{A(t) + B(t)} \right), \]

\( A(t) \) is the bid size, \( B(t) \) is the ask size and \( g(\cdot) \) is a cubic polynomial.

- **Assumptions:**
  - \( S(t) \) is a Lévy process,
  - \( S(t) \) is a martingale,
  - \( S^b(t) \) is in \( \mathbb{Z} \) and \( \theta(t) \in [0, 1) \).
The Optimal Liquidation Problem

- Efficient price process: \( S(t) = S^b(t) + \theta(t) \).
- Submitting a sell order at time \( t \) yields payoff \( S^b(t) = \lfloor S(t) \rfloor \leq S(t) \).
- **Goal:** Identify an optimal time \( \tau \) in \( [0, T] \) to sell the share and in turn to receive \( \lfloor S(t) \rfloor \), i.e.,

\[
V(t, s) = \sup_{t \leq \tau \leq T} E[\lfloor S(\tau) \rfloor | S(t) = s],
\]

for \( s \in \mathbb{R} \) and \( t \in [0, T] \), and \( \tau \in \mathcal{T} \), where \( \mathcal{T} \) is the set of stopping times with respect to \( \sigma(S(t))_{t \geq 0} \).
Trade/No-trade Regions

- “Trade” and “No-trade” region

\[ D = \{(t, s) \in [0, T] \times \mathbb{R} : V(t, s) = \lfloor s \rfloor \}, \]
\[ C = \{(t, s) \in [0, T] \times \mathbb{R} : V(t, s) > \lfloor s \rfloor \}. \]

- Liquidation time

\[ \tau_D = \inf \{ t \geq 0 | S(t) \in D \}. \]

**Proposition**
\[ \tau_D \in T \quad \text{and} \]
\[ V(t, s) = \mathbb{E}[\lfloor S(\tau_D) \rfloor | S(t) = s]. \]
Structural Properties of Value Function

**Proposition**

The function $V(t, s)$ satisfies the following properties:

(a) fix $t \in [0, T]$, then $V(t, s)$ is non-decreasing in $s$;

(b) fix $s \in \mathbb{R}$, then $V(t, s)$ is non-increasing in $t$;

(c) $V(t, s + z) = V(t, s) + z$ for all $s \in \mathbb{R}$, $t \in [0, T]$ and $z \in \mathbb{Z}$;

(d) $V(t, z) = z$ for all $t \in [0, T]$ and $z \in \mathbb{Z}$;

(e) $V(T, s) = \lfloor s \rfloor$ for all $s \in \mathbb{R}$. 
State Space Reduction

1. Property (c) shows that

\[ V(s, t) = \sup_{t \leq \tau \leq T} \mathbb{E}[S(\tau) | S(t) = s] = \sup_{t \leq \tau^\theta \leq T} \mathbb{E}[S(\tau^\theta) | S(t) = s], \]

where \( \tau^\theta \) are stopping times adapted to \( (\theta(t))_{t \geq 0} \).

2. Further, for \( \tau \in T \):

\[
\mathbb{E}[S(\tau) | S(t) = s] = \mathbb{E}[S(\tau) - \theta(\tau) | S(t) = s] \\
= s - \mathbb{E}[\theta(\tau) | S(t) = s].
\]

Hence, the problem \( V(t, s) \) is equivalent to

\[ V^\theta(t, u) = \inf_{t \leq \tau^\theta \leq T} \mathbb{E}[\theta(\tau^\theta) | \theta(0) = u], \]

and \( V(s, t) = s - V^\theta(t, s - \lfloor s \rfloor) \).
Optimal Liquidation based on Minimizing Imbalance

Define

\[ D^\theta = \left\{ (t, u) \in [0, T] \times [0, 1) : V^\theta(t, u) = u \right\}, \]
\[ C^\theta = \left\{ (t, u) \in [0, T] \times [0, 1) : V^\theta(t, u) < u \right\}. \]

**Proposition**

There exists a non-decreasing function \( w^* : [0, T] \rightarrow [0, 1] \) with \( w^*(T) = 1 \), such that \( D^\theta = \{ (u, t) \in [0, 1) \times [0, T) : u \leq w^*(t) \} \).
Trade/no Trade Regions

Jump Process, $\lambda(T) = 500$, $K = 0.4$, $\sigma = 0.005$

- Latency = 0T
- Timesteps: 10000
- States: 500
Sensitivity of the Trade Region

As the volatility of the price process $S(t)$ increases one can liquidate less aggressively (assuming no risk aversion).
Trading with Latency

- A trade triggered at time $t$ is executed at time $t + l$ for $l > 0$.
- Consider
  \[
  V^l(t, s) = \sup_{t \leq \tau \leq T - l} \mathbb{E}[S^b(\tau + l)|S(t) = s],
  \]
  where $\tau \in T$.
- Define the payoff function $G^l(s) = \mathbb{E}[S^b(l)|S(0) = s]$ for $s \in \mathbb{R}$, then, for $\tau \in T$,
  \[
  \mathbb{E}[S^b(\tau + l)|S(t) = s] = \mathbb{E}[[\mathbb{E}[S^b(\tau + l)|S(\tau)]|S(t) = s]
  = \mathbb{E}[G^l(S(\tau))|S(t) = s].
  \]
- Therefore
  \[
  V^l(t, s) = \sup_{t \leq \tau \leq T - l} \mathbb{E}[G^l(S(\tau))|S(0) = s].
  \]
Structural Properties

**Proposition**

The function $V^l(t, s)$ satisfies the following properties:

(a) fix $t \in [0, T]$, then $V^l(t, s)$ is non-decreasing in $s$;
(b) fix $s \in \mathbb{R}$, then $V^l(t, s)$ is non-increasing in $t$;
(c) $V^l(t, s + z) = V^l(t, s) + z$ for all $s \in \mathbb{R}$, $t \in [0, T]$ and $z \in \mathbb{Z}$;
(d) $V^l(T - l, s) = G^l(s)$ for all $s \in \mathbb{R}$.
Latency is Costly

**Proposition**

Fix $t \in [0, T], s \in \mathbb{R}$, then $V'(t, s)$ is non-increasing in $l$ for $l \in [0, T]$. 
Reducing the State Space

Analogous to the no-latency case one can show that $V^l(t, s)$ is equivalent to

$$V^{l, \theta}(t, u) = \inf_{t \leq \tau^\theta \leq T - l} \mathbb{E}[G^{\theta, l}(\tau^\theta) | \theta(t) = u],$$

where $G^{\theta, l}(u) = \mathbb{E}[\theta(l) | \theta(0) = u]$. 

![Payoff Function $G^{\theta, l}(\theta(t))$](image)
Trade/No-Trade Regions with Latency

The “trade region” is still connected, but the “no-trade” region does not need to be connected anymore:

**Proposition**

There exists a non-decreasing function \( w^*_i : [0, T] \rightarrow [0, 1] \) and a non-increasing function \( v^*_i : [0, T] \rightarrow [0, 1] \), with \( v^*_i \leq w^*_i \), \( w^*_i(t) = 1 \) for \( t \in [T - l, T] \) and \( v^*_i = 0 \) for \( t \in [T - l, T] \), such that

\[
D_{\theta,l} = \{(t, u) \in [0, T] \times [0, 1) : v^*_i(t) \leq u \leq w^*_i(t)\}.
\]
The red line shows the optimal policy without adjustment.
Discretization Approximation

- Knowing $V^l(t, s)$, resp., $V^{\theta, l}(t, u)$, is enough to identify good liquidation times.

- For general Lévy-processes no closed-form solutions exist, hence we rely on a time- and space-discretization.

- Let $N, E \in \mathbb{N}$. Define,

  $$k : [0, T] \rightarrow K = \{0, \ldots, N\}$$
  $$t \mapsto k(t) = \sup \{n \in \{0, \ldots, N\} \mid nT/N \leq t\},$$

  $$h : [0, 1) \rightarrow H = \{1, \ldots, E\}$$
  $$x \mapsto h(x) = \lfloor Ex \rfloor + 1.$$

- These mappings transform the original state space $[0, T] \times [0, 1)$ into a discrete state space with $(N + 1)E$ states.
Discretization Approximation cont.

- Consider the discrete-time, discrete-space version of $\theta(t)$, i.e.,
  \[ \tilde{\theta}(j(t)) = h(\theta(t)). \]

- Let $P_{\tilde{\theta}}$ denote the transition matrix of the homogenous Markov chain $\tilde{\theta}(n)$, i.e., the matrix with entries
  \[ p_{ij} = \mathbb{P}\left( \tilde{\theta}(n + 1) = j | \tilde{\theta}(n) = i \right), \]
  for $i, j \in \{1, \ldots, E\}$ and $n \in \{0, \ldots, N\}$.

- Approximate $V^{\theta,l}(t, u)$ and $D^{\theta,l}$ with
  \[ \tilde{V}_{E,N}^{\theta,l}(n, i) = \inf_{\tau \in \tilde{T}^{\theta,l}} \mathbb{E}[\tilde{G}^{\theta,l}(\tau) | \tilde{\theta}(n) = i], \]
  \[ \tilde{D}^{l,\theta}_{E,N} = \left\{ (n, i) \in K \times H : \tilde{V}_{E,N}^{\theta,l}(n, i) = \tilde{G}^{\theta,l}(i) \right\}, \]
Dynamic Program

- Bellman’s recursion:
  \[ V_{E,N}^{\theta,l}(n, i) = \max \left\{ \tilde{G}_{E,N}^{\theta,l}(i), \mathbb{E}[V_{E,N}^{\theta,l}(n + 1, \tilde{\theta}(n + 1)) | \tilde{\theta}(n) = i] \right\}, \]
  for \( i \in \{0, \ldots, N\} \) and \( n \in \{0, \ldots, N\} \).

- Conditional probability:
  \[ \mathbb{E}[V_{E,N}^{\theta,l}(\tilde{\theta}(n + 1), n + 1) | \tilde{\theta}(n) = i] = \sum_{k=1}^{E} p_{ik} V_{E,N}^{\theta,l}(n + 1, k). \]
Discretization Convergence

As $N \to \infty$ and $E \to \infty$ the boundary between trade and no-trade region converges to a smooth curve.
Empirical Backtesting
Set Up

- Backtesting on TAQ data for 5-years US treasury bonds for 21 days (July 2010).
- Assume that one lot is traded per minute (avoids trading on the same quote).
- Need a price model for

\[ S(t) = S^b(t) + \theta(t), \]

the bid price \( S^b(t) \) can be observed in TAQ data.
- Many possibilities to construct \( \theta(t) \) based on limit order book data.
- \( \theta(t) \) is the trade secret of many algorithmic trading companies.
Imbalance Process

- We use

\[ \theta(t) = g \left( \frac{B(t)}{A(t) + B(t)} \right) , \]

where \( A(t) \) (\( B(t) \)) is best ask (bid) size and \( g(\cdot) \) is a cubic polynomial with constraints \( g(0) = 0, g(0.5) = 0.5 \) and \( g(1) = 1 \) (leaving 1 degree of freedom).

- This transformation makes the stationary distribution almost uniform, which is necessary since \( S(t) \) is a Lévy process.
Empirical Evidence for Price Model

Empirical observed imbalance $\theta(t)$ **conditioned a trade occurs on the next quote** gives credential that traders use similar trading strategies as described.
Calibrate Price Model

- Assume $S(t) = S^b(t) + \theta(t)$ is a pure jump process.
- Jumps occur according a homogenous Poisson process with intensity parameter $\lambda = 355$ jumps per minute.
- A two-sided generalized Pareto distribution with shape parameter $K = 0.4$ and scale parameter $\sigma = 0.01$ provides a good fit of the jump distribution.
Optimal Stopping vs. TWAP Strategy

• The time-weighted average price (TWAP) strategy liquidates one share per minute independently of the state of the limit order book. (Only trades when spread=1).

• Consider residuals $R = S^b_\tau - S^b_0$, where $\tau$ is the stopping time from the optimal stopping problem $V(t,s)$ calibrated to a pure jump process $S(t)$.

• Compare 5,649 intervals of length 1 minute.

• Without latency the optimal liquidation strategy saves on average 26 $ per share, i.e., 1/3 of the spread (Spread is 78$ for 5 yrs US-treasury bonds):

\[
\begin{array}{c|c|c|c}
\text{Optimal policy vs. TWAP} & \mathbb{E}[R] & \sigma(\hat{R}) & P\text{-value} \\
26.26 $ & 49.14 $ & 2.64 \cdot 10^{-200}
\end{array}
\]
Realized Imbalances

Realized Imbalances TWAP

Realized Imbalances Optimal Stopping

Counts

Counts

θ(t)
Effect of Trading Horizon $T$
Cost of Latency

- **Cost of latency:**

\[
\text{COL} = \mathbb{E}[S_b(\tau^0 + l) - S_b(\tau^0)],
\]

where \(\tau^0\) is the stopping time induced by \(V(t, s)\).

- **Adjusted cost of latency:**

\[
\text{COL}_{adj} = \mathbb{E}[S_b(\tau^I + l) - S_b(\tau^0)],
\]

where \(\tau^I\) is the stopping time induced by the adjusted problem \(V^I(t, s)\).

- Note, we do not calculate the COL with respect to the TWAP strategy, but with respect to the **optimal strategy with no latency**.
The Cost of Latency cont.

- 10ms latency \(\approx\) 10$ per share.
- For latencies \(\geq 2000\)ms (i.e., 2 secs) the advantage of observing the limit order book diminishes (performance becomes similar to TWAP).
- Adjusting the liquidation policy brings only minor improvement in the performance.
Conclusions

• We consider an optimal stopping problem that depends on:
  • Information found in the order book;
  • Latency;
  • The time left to catch up with the TWAP algorithm.
• The solution comes in the form of a trade/no-trade regions in the imbalance process.
• We estimate model parameters with "Level II" trades and quotes data.
• We find that our optimal liquidation algorithm significantly outperforms a TWAP algorithm.
• We quantify the cost of latency.
• Future research:
  1. Modeling limit order executions.
  2. Game-theoretic considerations, i.e., prevent front-running of such a liquidation algorithm.
THANK YOU!