Optimal Trade Execution by Limit Orders under Coherent Dynamic Risk Measures

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Outline

• Strategies of low frequency (1min—1hour)
  – Trade execution under dynamic coherent risk
  – Optimal partition

• Strategies of high frequency (1sec—1min)
  – Simplified limit order book model
  – MDP and an index policy

• Numerical experiments on real data
  – Combinations of low frequency strategies and high frequency strategies.
## Trade Execution

### Before selling

<table>
<thead>
<tr>
<th>Stock</th>
<th>Facebook</th>
<th>Cash</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>$51.75</td>
<td></td>
</tr>
<tr>
<td>Shares</td>
<td>250,000</td>
<td>$0</td>
</tr>
</tbody>
</table>

### After selling

<table>
<thead>
<tr>
<th>Stock</th>
<th>Facebook</th>
<th>Cash</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>$48.93</td>
<td></td>
</tr>
<tr>
<td>Shares</td>
<td>0</td>
<td>$12M</td>
</tr>
</tbody>
</table>

The price drops during a large transaction.

To reduce the cost:

**Divide the large asset into smaller ones and sell them sequentially**
• $x_0$: The initial shares of the stock
• $[0, D]$: Time interval.
• $N$: The number of transactions.
• $\tau = \frac{D}{N}$: The duration of each transaction.
• $n_k \geq 0$: The size of the transaction in $[(k - 1)\tau, k\tau], k = 1 \ldots N$
\[
\sum_{k=1}^{N} n_k = x_0
\]

- $S_k$: The market price of the stock at time $(k - 1)\tau$
- $S_k = S_{k-1} + \xi_k - gn_k$, for $k = 1, \ldots, N$
- $\xi_1, \ldots, \xi_N$: i.i.d. random vectors on $(\Omega, F, P)$
- $\tilde{S}_k = S_{k-1} - h \frac{n_k}{\tau}$: The average transaction price of $n_k$
Notations and Models

Implementation Shortfall:
\[ C(\Delta) = S_0 x_0 - \sum_{k=1}^{N} \tilde{S}_k n_k \]
\[ \Delta = (n_1, n_2, \ldots, n_N) \]

Optimal execution:
\[ \min \rho(C(\Delta)) \]
\[ \sum_{k=1}^{N} n_k = x_0, \quad n_k \geq 0, \]
\[ n_k \text{ is measurable w.r.t. } \sigma(S_0, \ldots, S_{k-1}) \]

<table>
<thead>
<tr>
<th>( \rho(\cdot) )</th>
<th>References</th>
<th>Time Consistent</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(\cdot) )</td>
<td>Bertsimas&amp;Lo,1998</td>
<td>Yes</td>
<td>Closed form</td>
</tr>
<tr>
<td>( E(\cdot) + \lambda \text{Var}(\cdot) )</td>
<td>Almgren&amp;Criss,1999,2000, Almgren&amp;Lorenz,2007,2011</td>
<td>Yes</td>
<td>closed form if static/numerical solution if adaptive</td>
</tr>
<tr>
<td>( CVaR(\cdot) )</td>
<td>Moazeni et al, 2012</td>
<td>No</td>
<td>Linear policy</td>
</tr>
<tr>
<td>Dynamic Coherent Risk</td>
<td>This talk</td>
<td>Yes</td>
<td>Shown later</td>
</tr>
</tbody>
</table>
Other models and methods

- Schied and Schoneborn, 2009
- Alfonsi, Fruth and Schied, 2010
- Gatheral, 2010
- Predoiu, Shaiket and Shreve, 2010
- Shied, Schoneborn and Tehranchi, 2010
- Alfonsi, Schied and Slynko, 2012
- Gatheral and Schied, 2011
- Forsyth, Kennedy, Tse and Windcliff, 2011
- Guo and Zervos, 2012
- Obizhaeva and Wang, 2013
- Moallemi and Saglam, 2013
- Tsoukalas, Wang and Giesecke, 2010
- ........
Coherent risk measure

Example

Conditional Value at Risk (CVaR):

$$\text{CVaR}_\alpha(Z) = \inf_{r \in \mathbb{R}} \left\{ r + \frac{1}{1 - \alpha} \mathbb{E}[(Z - r)_+] \right\}.$$ 

Here, $\alpha \in [0, 1]$ is a risk aversion parameter.
Conditional risk mapping of CVaR

Ruszczynski and Shapiro, 2006

Example

Suppose $F'$ is a sigma algebra of $\Omega$. The conditional risk mapping of CVaR is:

$$\text{CVaR}_\alpha(Z|F') = \inf_{r \in \mathbb{R}} \left\{ r + \frac{1}{1 - \alpha} \mathbb{E}[(Z - r)_+|F'] \right\}. $$

Here, $\alpha \in [0, 1]$ is a risk aversion parameter.

$\text{CVaR}_\alpha(Z|F')$ is a $F'$ measurable random variable.
Dynamic Risk Measure (Ruszczynski, 2009)

\[ F_1 \subset F_2 \subset \cdots \subset F_N \]

\( F_k \): The filtration generated by \( \{ S_0, S_1, \ldots, S_{k-1} \} \)

\[
\begin{align*}
&\text{CVaR}_\alpha(Z | F_N) \\
&\text{CVaR}_\alpha(\text{CVaR}_\alpha(Z | F_N) | F_{N-1}) \\
&\text{CVaR}_\alpha(\text{CVaR}_\alpha(\text{CVaR}_\alpha(Z | F_N) | F_{N-1}) | F_{N-2}) \\
\end{align*}
\]

\[
\rho_{1,N}(Z) = \text{CVaR}_\alpha(\cdots \text{CVaR}_\alpha(\text{CVaR}_\alpha(Z | F_N) | F_{N-1}) \cdots | F_1)
\]

\[
\min \rho_{1,N}(C(\Delta))
\]

\[
\sum_{k=1}^{N} n_k = x_0, \quad n_k \geq 0, \\
n_k \text{is measurable w.r.t. } F_k
\]
A closed-formed optimal solution

Theorem:

• If $\xi_k \sim N(0, \tau \sigma^2)$ and i.i.d., the optimal solution is static:

$$n_k = \left( \frac{x_0}{N^*} + \frac{(N^* - 2k + 1)a}{2} \right)_+ \quad \text{for } k = 1, \ldots, N,$$

where $N^* \equiv \min \left\{ N, \left[ \frac{-1 + \sqrt{1 + 8x_0/a}}{2} \right] \right\}$

$$a = \frac{\sqrt{\tau \sigma CVaR_\alpha(N(0,1))}}{2h \frac{\tau}{\tau - g}}$$

• If $\xi_k$’s are independent and $h$ and $g$ changes with time, the optimal solution is also static but in a more sophisticated form.

• In the multi-asset case, if $\xi_k \sim N(0, \tau \Sigma)$, the optimal solution is also static and can be solved by SOCP. If $\xi_k$’s are independent, we can solve by saddle-point+SA.
Sparse Solution V.S. Long Tail

A Simulated Example: $N = 20, x_0 = 10000, \tau = 19.5\text{mins}, h = 1, \sigma = 1, 9:30\text{am-4:00pm}$

$$\min_{\Delta} E(C(\Delta)) + \lambda \text{Var}(C(\Delta))$$
$$\lambda = 0.0001$$

$$\mathbb{E}C(\Delta) = 3.6231e + 005 \text{ ($\$$)}$$

Same expected shortfall

$$\min_{\Delta} \rho_{1,N}(C(\Delta))$$
$$\alpha = 0.02$$
Sparse Solution V.S. Long Tail

A Simulated Example: \( N = 20, x_0 = 10000, \tau = 19.5 \text{mins}, h = 1, \sigma = 1, 9:30\text{am-4:00pm} \)

\[
\min_{\Delta} E(C(\Delta)) + \lambda Var(C(\Delta)) \quad \lambda = 0.0001
\]

\[\text{EC}(\Delta) = 3.6231e+005 \ ($)\]

Same expected shortfall

\[
\min_{\Delta} \rho_{1,N}(C(\Delta)) \quad \alpha = 0.02
\]
Sparse Solution V.S. Long Tail

\[
\min_{\Delta} E(C(\Delta)) + \lambda \text{Var}(C(\Delta))
\]

\[
\min_{\Delta} \rho_{1,N}(C(\Delta))
\]
• How to sell $n_k$ in $[(k-1)\tau, k\tau]$?
  – Market Orders: large $h$, guaranteed to be executed by $k\tau$.
  – Limit Orders: small $h$, not guaranteed.
  – Both: use limit orders before $k\tau$ and market order at $k\tau$. 

![Bar chart showing Stage (k) vs. count]
Limit order book

- Rosu, 2009
- Cont, Stoikov & Talreja, 2011
- Hult & Kiessling, 2010
- Maglaras & Moallemi, 2011
- Maglaras, Moallemi & Hua, 2012
- Cont & de Larrad, 2013
- …..
To sell $n_k$ shares in $[(k - 1)\tau, k\tau]$ and take actions only at time $(k - 1)\tau + \frac{\tau t}{T}, t = 0,1,2, ... T - 1$

- $\delta$: tick size, e.g. 1 cent
- $p_t$: best ask price
- $p_t - \delta$: best bid price
- $a_t$: the total volume of sell limit orders at the best ask price
- $b_t$: the total volume of buy limit orders at the best bid price

Assumption 1: The bid-ask spread is always one tick. (Cont and de Larrard, 2013)
A simplified limit order model

- $l_t$: The volume of our order in the book (not counted in $a_t$)
- $r_t$: The volume arrived the earlier than us at the best ask
- $x_t$: The remaining volume.

Assumption 2: We can only have one order in the book and it must be on the best ask price

State variables: $p_t$, $a_t$, $b_t$, $l_t$, $r_t$, $x_t$
A simplified limit order model

Action space:
- \( W \): wait and do nothing
- \( l \): cancel current order and place a new order of size \( l \) at the best ask price

If the current order is not at the best ask price, \( l \) is the only option. (Assumption 2)
A simplified limit order model

A simplified dynamic of LOB:
1. “Aggregated” bid and ask market orders $A_t$ and $B_t$ arrive.
2. “Aggregated” bid and ask limit orders $\alpha_t$ and $\beta_t$ arrive.
3. The ask and bid orders are canceled at rates $c_t$ and $d_t$ respectively.
If we choose “$W$”

**Case 1:** $A_t < a_t + l_t$, $B_t < b_t$

- $p_{t+1} = p_t$

Market orders come
If we choose “$W$”

**Case 1:** $A_t < a_t + l_t$, $B_t < b_t$

- $p_{t+1} = p_t$

Limit orders come
Case 1: $A_t < a_t + l_t$, $B_t < b_t$

- $p_{t+1} = p_t$
- $l_{t+1} = (l_t - (A_t - a_t)_+)_+$
- $a_{t+1} = [a_t - \max(\min(A_t, a_t), A_t - l_t) + \alpha_t](1 - c_t)$
- $b_{t+1} = (b_t - B_t + \beta_t)(1 - d_t)$
- $r_{t+1} = (a_t - A_t)_+(1 - c_t)$
- $x_{t+1} = x_t - \min(l_t, (A_t - \alpha_t))$

If we choose “$W$”
If we choose “\( W \)"

**Case 2:** \( A_t \geq a_t + l_t, \ B_t < b_t \)

Market orders come
If we choose “$W$”

**Case 2:** $A_t \geq a_t + l_t, \ B_t < b_t$

- $p_{t+1} = p_t + \delta$
- $(a_{t+1}, b_{t+1}) \leftarrow \text{pdf } \phi(a, b)$
- $r_{t+1} = a_{t+1}$
- $l_{t+1} = 0$
- $x_{t+1} = x_t - l_t$

Cont & de Larrad, 2013
If we choose “$W$”

**Case 3:** $A_t < a_t + l_t$, $B_t \geq b_t$

Market orders come
If we choose “$W$”

**Case 3:** $A_t < a_t + l_t, \ B_t \geq b_t$

- $p_{t+1} = p_t - \delta$
- $(a_{t+1}, b_{t+1}) \leftarrow pdf \ \phi(a, b)$
- $r_{t+1} = a_{t+1}$
- $l_{t+1} = 0$
- $x_{t+1} = x_t - \min(l_t, (A_t - \alpha_t)_+)$

![Diagram showing the case 3 conditions](image)
If we choose “W”

Case 4: \( A_t \geq a_t + l_t, \ B_t \geq b_t \)

Market orders come
If we choose \( W \)

**Case 4:** \( A_t \geq a_t + l_t, \; B_t \geq b_t \)

- \( p_{t+1} = p_t \)
- \((a_{t+1}, b_{t+1}) \leftarrow pdf \; \phi(a, b)\)
- \( r_{t+1} = a_{t+1} \)
- \( l_{t+1} = 0 \)
- \( x_{t+1} = x_t - l_t \)
If we choose “$l$”

• If we cancel the current order and place a new order with a size of $l$ in the best ask price

• The state transition is the same except that:
  – $l_t$ is replaced by $l$
  – $r_t$ is replaced by $a_t$
Markov Decision Process

\[
\max \sum_{t=0}^{T-1} E\{R(p_t, a_t, b_t, l_t, r_t, x_t, Action_t)\} + E\{M(p_T, x_T)\}
\]

- \(Action_t \in \{W, l \in [0, x_t]\}\)
- \(R(p_t, a_t, b_t, l_t, r_t, x_t, Action_t)\): The reward in stage \(t\)
- \(M(p_T, x_T)\): The final reward by selling the remaining volume \(x_T\) with a single market order.

Proposition:
If \(t = T - 1\), \(A_t \sim \text{Exp}(\mu_A)\) and \(B_t \sim \text{Exp}(\mu_B)\), the best \(l\) is either 0 or \(x_t\)
Index policy

- When making decision, assume $t = T - 1$
- The total reward:
  $$I(p_t, a_t, b_t, l_t, r_t, x_t) = E\{R(p_t, a_t, b_t, l_t, r_t, x_t) + M(p_{t+1}, b_{t+1}, x_{t+1})\}$$
- Choose the largest one in
  - Wait: $I(p_t, a_t, b_t, l_t, r_t, x_t)$
  - Cancel: $I(p_t, a_t, b_t, 0, a_t, x_t)$
  - Place all: $I(p_t, a_t, b_t, x_t, a_t, x_t)$
Numerical experiments

Sell 1000 shares of DAL in “5-min” starting at 78 different times in every trading day in July 2010:
• Sell with 10 market orders with equal size (100 share)
• Sell with 10 limit orders with index rule (T=10 in MDP)

Use NYSE limit order data to rebuild the book in every millisecond in July, 2010

\[
\tilde{S}_k = S_{k-1} - h \frac{n_k}{\tau}
\]

Market Order: \( h = 3.62 \times 10^{-4} \)
Limit Order: \( h = 9.51 \times 10^{-5} \)
Numerical experiments

\[ n_k = \frac{x_0}{N} \text{ for } k = 1, \ldots, N. \]
\[ n_k = \frac{2\sinh(\lambda \tau/2)}{\sinh(\lambda T)} \cosh(\kappa (T - (k - 0.5) \tau)) x_0 \text{ for } k = 1, \ldots, N \]
\[ n_k = \left( \frac{x_0}{N^*} + \frac{(N^*-2k+1)\alpha}{2} \right)_+ \text{ for } k = 1, \ldots, N, \text{ where } N^* = \min \left\{ N, \left[ -\frac{1 + \sqrt{1 + 8x_0/\alpha}}{2} \right] \right\} \]

\{Ber&Lo, Alm&Chr, Lin et al\} ∩ (Market Order, Limit Order)

- Sell 78000 shares of DAL every day from 9:30am to 4:00pm in July, 2010.
- Use NYSE limit order data to rebuild the book in every millisecond
- \( N = 78, \tau = 5\text{min} \)
- Use moving window (5 days) to estimate \( h, \sigma, \mu_A \text{ and } \mu_B \)

<table>
<thead>
<tr>
<th>Execution</th>
<th>Market Orders with Equal Size</th>
<th>Limit Order with Index Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partition</td>
<td>Ber&amp;Lo (( \lambda ) tuned)</td>
<td>Ber&amp;Lo (( \lambda ) tuned)</td>
</tr>
<tr>
<td>Mean</td>
<td>5502$</td>
<td>703$</td>
</tr>
<tr>
<td>CVaR_{0.66}</td>
<td>12185$</td>
<td>1555$</td>
</tr>
</tbody>
</table>
Future work

- From one queue to multiple queues
- Solve MDP by ADP
- When the price is not Markovian