Modelling High Frequency Data by Hawkes Processes

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Purpose: give a model for an order book

Buy and sell orders did not arrive at continuous times, counterparts do not meet them at any time, hence, the fluctuations of the stock market cannot evolve continuously.

High frequency financial data have a flagrant discontinuity.

- The idea to use point processes to describe such irregularities appears as an evidence.
The family of stochastic processes that we will use in our analysis, the so-called Hawkes processes, was introduced in the seventies by A. Hawkes.

A Hawkes process is a generalization of a standard point process. The initial purpose of this process was to model earthquake occurrences.
They already found applications in various fields.

We refer, among others, to Ogata for statistical earthquake modeling, to Chavez-Demoulin et al. and Errais et al. for risk analysis, Aït-Sahalia et al. for the so-called financial contagion, to Bacry et al. for high-frequency financial data and to Muni Tokes for order book modeling.
In our model we will consider two types of orders:
- marker orders
- limited orders

And two types of situations
- the model with only one agent
- the model with several agents
A market order will immediately be executed at the best available price and at the relevant time.

A buy market order will be executed to the best ask price, while sell market order will be executed to the best bid price. The best ask (bid) price is understood at the lowest price at which an agent can buy (sell) an asset.

The volume offered at the best ask price need to be completely bought to see the best ask (bid) price moving upward (downward) from one tick.
A limited order is an order to buy (sell) an asset to a specific price, lower (higher) than the best ask (bid) price.

Then, a new offered volume inside the bid-ask spread will move the price downward or upward depending if it is proposed on ask side or bid side.
A market order will decrease the amount of shares available on the market (we call it "the volume"), while a limited order inside the bid-ask spread will increase them.

**Figure**: Schematic representation of an Order Book. In red volume proposed for sell, in blue volume proposed for buy. Left: initial state. Middle: same order book after a buy market order which move upward the best ask with one tick. Right: a new buy limited order inside the bid-ask spread which move upward the best bid.
Thus, the exchanged volume affects the price and the financial fluctuation.

The time between two transactions and the volume are dependent since one produces the other one. Several papers related to the theory of market impact try to deals with this aspect, see e.g. E. Moro et al., or Z. Eisler et al.

Our purpose is to incorporate this information concerning the variable ”volume of orders” in our model.

To do this, we will employ the marked Hawkes process, their mark corresponding to the volume of each transaction.
Let $N(t)$ a $d$-dimensional point process, $N = (N_1, N_2, \ldots, N_d)$, $N_i(t_i), 1 \leq i \leq d$ the cumulative number of events for the $i^{th}$ component at time $t_i$.

Note that this process $N_i(t_i), i = 1, .., d$ must be non-decreasing and right continuous, and take only non-negative integer values.
An important characteristic of the process $N$ is its conditional intensity with respect to the filtration considered and this conditional intensity is given by

$$\lambda(t|\mathcal{F}_t) = \lim_{\delta t \searrow 0} \frac{1}{\delta t} \mathbb{E} [N(t + \delta t) - N(t)|\mathcal{F}_t].$$

(1)

The $\mathcal{F}_t$-intensity characterizes the evolution of the process $N(t)$ with respect to this past history $\mathcal{F}_t$.

We could interpret that as the conditional probability at time $t$ to observe a new event at next time $t + \delta t$. 
The Hawkes process is a particular class of the point processes. This class is characterized by the intensity function. More precisely, the intensity of a multivariate \(d\)-dimensional Hawkes process is usually defined by

\[
\lambda_i(t | \mathcal{F}_t) = \mu_i + \sum_{j=1}^{d} \nu_{ij} \int_{-\infty}^{t} h_i(t-s)N_j(ds),
\]

(2)

where \(h = \{h_i\}_{i=1,\ldots,d}\) is the so-called decay kernel and satisfies \(h_i(t) \geq 0\) for all \(i\).
The parameters \( \{ \nu_{ij} \}_{i,j=1,\ldots,d} \) are referred to as the branching coefficients, they quantify the ability of an event \( i \) to trigger an event of type \( j \). Thus, we see the mutually exciting structure of the process since event \( j \neq i \) can affect the conditional intensity \( \lambda_i \).

Self-exciting part is of course the case \( j = i \), past and current event will induce a response in their own intensity process and therefore, on the corresponding point process.

The constants \( \{ \mu_i \}_{i=1,\ldots,d} \) are understood as the rate of instantaneous events.
The purpose of this work is to incorporate the volume of the orders in our model.

we will use multivariate Hawkes process but we will incorporate the marks in the intensity expression (2).

A mark is an additional value attached to each point and brings some new information about the points.

Consequently, we will have marked intensities, denoted in the sequel \( \lambda(t, v_t|\mathcal{F}_t) \) where \( v_t \) represents the mark and will model the volume of the orders at time \( t \).
The marked intensity (see e.g. Daley and D. Vere-Jones 03) takes the form,

\[ \lambda_i(t, v | \mathcal{F}_t) = \mu_i + \sum_{j=1}^{d} \nu_{ij} \int_{(-\infty, t) \times \mathbb{R}^+} h_i(t-s)g_j(v)N_j(ds \times dv). \]  

(3)

Now, \( \mathcal{F}_t \) is the history of arrival time and corresponding mark, \{\( t_i, v(t_i) \}\).

The function \( g_j, j = 1, \cdots, d \) is the so-called impact function of marks

in other words, it characterizes the impact of the volume on the financial asset fluctuation.
It has to satisfy the condition \( \int_{\mathbb{R}^d} g(v)f(v)dv = 1 \) where \( f \) is the density of the vector 
\( V = (V_1, \cdots, V_d), V_i = (v_i(1), \cdots, v_i(t), \cdots, v_i(T)). \)
Most common choices for the impact function are,

\[ \tilde{g}(x) = x^\alpha, \quad \alpha > 0, \quad \text{and} \quad \tilde{g}(x) = \exp(\alpha x), \quad \alpha > 0. \]

- the first choice corresponds to an impact in power law, and the second one means an exponential impact of the volume on the asset price.
We used the notation $\tilde{g}$ because we need to normalize the function in order to respect condition $\int_{\mathbb{R}^d} g(v)f(v)dv = 1$. Then the corresponding normalized versions are:

$$g(x) = \frac{x^\alpha}{\mathbb{E}X}, \quad \alpha > 0,$$

and

$$g(x) = \frac{\exp(\alpha x)}{\mathbb{E}X}, \quad \alpha > 0 \quad (4)$$

where $X$ is a random variable with density $f$. 
Without the impact function, or in other words without taking into account the mark, the intensity’s increase depends only on the time duration between two events.

Using marked Hawkes process, the intensity increases with respect to the arrival time of events but also with respect to the mark value. So intensity, and finally the price, will be influenced by the trading volume.

The market impact depends on the degree of excitation, but not only on the frequency of the arrival of significant events. It is also affected by the amount of the exchanges at each moment. It then natural to use marked point process instead of unmarked process.
The kernel \( h_i(t - s), \ i = 1, 2, ..., d \) gives us the probability that at time \( s < t \), an event of type \( j \) to trigger an event \( j \) at time \( t \). Notice that as for the non marked and univariate case, we simply have,

\[
\lambda_i(t, \nu|\mathcal{F}_t) = \mu_i + \sum_{j=1}^{d} \nu_{ij} \sum_{j=1}^{d} h_i(t - t_{k}^{(j)}) g_j(\nu(t_k^{(j)})).
\]
- an important volume of trading does not affect market in same manner as a small volume does.

One of the most popular stylized facts, the so-called cluster of return, implies that big changes are usually followed by big changes.

It is therefore natural to think that a large exchanged volume could trigger many trades, exciting the market.
The log-return of EUR/USD in “deci”-seconds sampling

**Figure**: EUR/USD log-return from 14:00:00 to 14:10:00, 06.02.2012 in decisecond sampling.
The model

Notation: \( a \) = best ask price, \( b \) = best bid price, + for upward jump and - for downward jump

\[
\begin{align*}
\lambda_{a,+}(t) &= \mu_{a,+} \\
+ \sum_{j = a+, b+} \nu_{a,+j,+} \int_{(-\infty,t) \times \mathbb{R}} h_{a,+}(t-s) g_j(v) N_j(ds \times dv) \\
\lambda_{a,-}(t) &= \mu_{a,-} \\
+ \sum_{j = a-, b-} \nu_{a,-j,-} \int_{(-\infty,t) \times \mathbb{R}} h_{a,-}(t-s) g_j(v) N_j(ds \times dv)
\end{align*}
\]

(5)
Bid:

\[
\begin{cases}
\lambda_{b,+}(t) = \mu_{b,+} \\
+ \sum_{j=a+,b+} \nu_{b,+,j,+} \int_{(-\infty,t)\times\mathbb{R}} h_{b,+}(t-s)g_j(v)N_j(ds \times dv)
\end{cases}
\]

\[
\begin{cases}
\lambda_{b,-}(t) = \mu_{b,-} \\
+ \sum_{j=a-,b-} \nu_{b,-,j,-} \int_{(-\infty,t)\times\mathbb{R}} h_{b,-}(t-s)g_j(v)N_j(ds \times dv).
\end{cases}
\]

(6)
The model is univariate, corresponding to only one agent.

the quantity $\lambda_{a,+}$ represents the intensity associated with the ask price when this price had an upward move.

Similarly are interpreted the intensities $\lambda_{a,-}, \lambda_{b,+}$ and $\lambda_{b,-}$.

The scalar $\nu_{a,+},b,+\ $ quantifies the interaction between the ask price and the bid price (in the upward jump case).
If one tries to incorporate some dependence between the assets prices, and to have for example the Epps effect, and/or the lead-lag effect, one a priori needs to extend the model to a multivariate form.

we assume only two types of interaction : the dependence for upward jumps in ask side in both assets, and the same for the downward jumps.
Mathematically: Writing $\lambda^{(i)}_{a,+}(t)$ for the intensity of the asset $i = 1, \cdots, d$ in the upward jump case and $\lambda_j(t)$ as in (6), asset $i$
The univariate model

The multivariate model

Bid\(^{(i)}\) :

\[
\begin{align*}
\lambda^{(i)}_{b,+}(t) &= \lambda_{b,+}(t) \\
&+ \sum_{j \neq i=1}^{d} \nu^{(j)}_{b,+} \int_{(-\infty, t) \times \mathbb{R}} h_{b,+}(t-s) g^{(j)}_{b,+}(v) N^{(j)}_{b,+}(ds \times dv) \\
&+ \sum_{j \neq i=1}^{d} \nu^{(j)}_{b,-} \int_{(-\infty, t) \times \mathbb{R}} h_{b,-}(t-s) g^{(j)}_{b,-}(v) N^{(j)}_{b,-}(ds \times dv) \\
&+ \sum_{j \neq i=1}^{d} \nu^{(j)}_{b,+} \int_{(-\infty, t) \times \mathbb{R}} h_{b,+}(t-s) g^{(j)}_{b,+}(v) N^{(j)}_{b,+}(ds \times dv).
\end{align*}
\]
the interactions that we take into account:

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**Table:** 'x' if true, '-' if false. \(i \neq j = 1, \cdots, d\). It is nothing else than the branching matrix \(\nu\). As we can see, in the case of two assets, we have for each quantity, four components, itself to produce the auto-excitation effect and two other, the corresponding quantities in the other asset, and bid with similar jump on the two assets. Up left and bottom right quart correspond to monovariate case \(i\) and \(j\).
The most natural estimation is given by the maximum of likelihood method.

Let $\Theta$ be the parameters set which depend on mark distribution $f = (f_1, \cdots, f_d)$. Also consider the impact function $g = (g_1, \cdots, g_d)$, the decay kernel $h = (h_1, \cdots, h_d)$, the branching matrix, $\nu = \{\nu_{ij}\}_{i,j=1,\cdots,d}$ and $\mu = (\mu_1, \cdots, \mu_d)$. Let $I = [T-, T+]$ be interval containing all arrival times.
With these notation, the likelihood function is given by:

\[ L(\{t_i, v_i\}; \Theta) = \prod_{j=1}^{d} \int_{I \times \mathbb{R}} \lambda_j(t, v(t)|\mathcal{F}_t) N_j(dt \times dv) \exp(-\Lambda_j(T^+)), \]

where \( \Lambda_j(T) \) is the compensator, or integrated intensity given by

\[ \Lambda_j(T) = \int_{-\infty}^{T} \lambda_j(t, v|\mathcal{F}_t) dt \times dv, \quad j \in \{1, \cdots, d\}. \]
The discrete version of log-likelihood is then expressed as follows:

\[
\log L(\{t_i, v_i\}; \Theta) = \sum_{j=1}^{d} \sum_{k=1}^{N} \log \lambda_j(t_k, v(t_k)|\mathcal{F}_{t_k}) - \sum_{j=1}^{d} \Lambda_j(T^+).
\]

The maximization of log-likelihood could be done using a classical optimization algorithm.
When a limited order to buy some shares is passed inside the quotes, it increases the bid price (upward jump on bid side). Conversely, when an limited order appears bellow the ask price, it decreases the ask price (downward jump on ask side). When a buy marked order takes all available shares, the ask price increases to the next limit available in the limited order book, (upward jump on ask side). Reciprocally, when a sell market order take all available shares of the first limit, bid price below to the next limit available on the order book (downward jump on bid side).
Quantity offered by a new limited order was greater than the quantity offered by an ‘old’ limited order which becomes the new bid or ask price.

**Figure**: Cumulative distribution in log scale of ask volume in a case of upward jump (blue circle), ask volume in a case of downward jump (blue cross), bid volume in a case of upward jump (red circle), bid volume in a case of downward jump (red cross) on EUR/USD (left) and EUR/GBP (right), from January 30, 2012 00:00:00 to March 09, 2012 21:59:00. Dashed line, gaussian fit; solid line, exponential fit.
there are some characteristics of very high frequency financial data

For each study of these quantities, we will compare the empirical result, the best fit obtained with an appropriate law and the realization given by the model.

Let $X(t) = \log p(t)$, the log price of an asset. The signature plot corresponds to the volatility evolution on $[0, T]$ with respect to frequency,

$$\nabla X(\tau) = \frac{1}{T} \sum_{n=0}^{\lfloor T/\tau \rfloor} (X((n+1)\tau) - X(n\tau))^2$$

(8)

so more the lag $\tau$ is large, more this quantity is decreasing in power law.
The explanation of that signature is quite simple: at very high frequency, data arrives at more or less distant time, asset prices are discontinuous and show jumps, hence, volatility tend to increase with the frequency because of these jumps.

Conversely, the fluctuations observed at large scale tend to smooth evolutions, hence, volatility tends to decrease until their equilibrium value. This effect is plotted using mid price quote by averaging week by week.
**Figure**: Signature plot of EUR/USD in black and in red for the simulation, fit in power law (green), $\tau$ varying for 0 to 100 seconds.
The Epps effect is the evolution of the correlation between two financial assets with respect to the frequency.

As frequency increases, the correlation tends to vanish. This is not very complicated to realize, if the orders arrive at discontinuous time. Even if we have two very liquid assets, it will be not likely to observe their a correlated variation at the same milisecond.
\rho_{1,2}(\tau) = \frac{\text{C}_{o_{1,2}}(\tau)}{\sqrt{\text{V}_{X_1}(\tau)\text{V}_{X_2}(\tau)}}

where \text{V}_{X}(\tau) is define like in (8) and the empirical covariation \text{C}_o is given by,

\text{C}_{o_{1,2}}(\tau) = \frac{1}{T} \sum_{n=0}^{[T/\tau]} (X_1((n+1)\tau) - X_1(n\tau))(X_2((n+1)\tau) - X_2(n\tau)).
This effect is plotted using mid price quote by averaging week by week.

**Figure**: Epps effect on the two considered currency foreign exchange data, EUR/USD and EUR/GBP, in black and in red for the simulation, corresponding fit in power law, \( \tau \) varying for 0 to 100 seconds.