Information and trading targets in a dynamic market equilibrium

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Motivation

- Many types of investors trade in financial markets with many different motives
  - Information
  - Portfolio rebalancing

- In equilibrium, market liquidity and information revelation depend on behavior and beliefs of all investors
  - In markets with adverse selection, behavior of trading noise is important

- First paper to study a market microstructure equilibrium with both dynamic order splitting and dynamic informed trading
  - Two types of trading noise: i.i.d. randomness from noise traders and autocorrelated randomness from portfolio rebalancing
Questions

- Effect of optimized rebalancing on...
  - Liquidity and information revelation?
  - Aggregate order flow dynamics?

- Nature of optimal rebalancing:
  - Smoothing of price impact?
  - Importance of sunshine trading?
  - Strategic interactions between rebalancer and informed orders?
Literature

- Optimal order-splitting models:
  - Minimize the expected cost of trading a fixed number of shares over $N$ periods
  - Price impact of order flow is an exogenously assumed function

- Equilibrium models:
  - Price impact of order flow is determined in equilibrium along with order flow dynamics
  - Single- and multiple-informed investors
  - Optimizing rebalancers:
    - Admati and Pfleiderer (1988) — short-lived information and one-period rebalancing
    - Seppi (1990) — parameter restrictions
    - Degryse, de Jong, and van Kervel (2014) — short-lived information, deterministic order-splitting strategy
HAPPY 30TH BIRTHDAY
KYLE (1985) MODEL!

... AND ALSO GLOSTEN & MILGROM (1985)!
Results

- Characterize a Bayesian Nash linear equilibrium
- Develop an algorithm for computing the equilibrium
- Numerical results
  - Order flow autocorrelation
  - $U$-shaped expected orders, rebalancer order flow volatility, price volatility
  - $S$-shaped price impact over the trading horizon
- Properties of equilibrium rebalancing
  - Learning
  - Negative correlation with informed order flow over time
  - Sunshine trading is relatively small
Model (1)

- Discrete-time model (Kyle 1985) with trading times \( n = 1,\ldots,N \)
- Four types of investors
  - **Rebalancer** with trading target \( \tilde{a} \sim N(0, \sigma_{\tilde{a}}^2) \). Submits orders \( \Delta \theta_n^R \)
  - **Insider** knows terminal stock value \( \tilde{v} \sim N(0, \sigma_v^2) \). Submits orders \( \Delta \theta_n^I \)
  - **Noise traders** trade Brownian motion increments \( \Delta w_n \sim N(0, \sigma_w^2 \Delta) \) where \( \Delta > 0 \) is the time step
  - **Market makers** set prices \( p_n \) based on observing aggregate order flow
    \[
    y_n = \Delta \theta_n^R + \Delta \theta_n^I + \Delta w_n
    \]
- \( \tilde{a} \) and \( \tilde{v} \) are jointly normal (correlation \( \rho \)) and independent of \( w \)
- All investors are all risk-neutral
- Prices are determined by the competitive zero-expected-profit condition for market makers:
  \[
  p_n = \mathbb{E}[\tilde{v} | \sigma(y_1, \ldots, y_n)]
  \]
Model (2)

- **Insider:**
  - Chooses order $\Delta \theta^I_n$ at time $n$ using the information $\sigma(\tilde{v}, y_1, \ldots, y_{n-1})$
  - Solves the optimization problem

\[
\max_{\Delta \theta^I_k \in \sigma(\tilde{v}, y_1, \ldots, y_{k-1})} \sum_{k=1}^{N} (\tilde{v} - p_k) \Delta \theta^I_k \bigg| \sigma(\tilde{v})
\]

- **Rebalancer:**
  - Chooses order $\Delta \theta^R_n$ at time $n$ using the information $\sigma(\tilde{a}, y_1, \ldots, y_{n-1})$
  - Solves the optimization problem (note the terminal constraint)

\[
\max_{\Delta \theta^R_k \in \sigma(\tilde{a}, y_1, \ldots, y_{k-1})} -\sum_{k=1}^{N} (\tilde{a} - \theta^R_{k-1}) \Delta p_k \bigg| \sigma(\tilde{a})
\]
Bayesian Nash Equilibrium (BNE)

A collection of functions \( \{\theta^I_n, \theta^R_n, p_n\} \) is a BNE if:

- given \( \{\theta^R_n, p_n\} \), the strategy \( \theta^I_n \) solves the insider’s problem
- given \( \{\theta^I_n, p_n\} \), the strategy \( \theta^R_n \) solves the rebalancer’s problem
- given \( \{\theta^I_n, \theta^R_n\} \), the pricing rule \( p_n \) satisfies

\[
p_n = \mathbb{E}[\tilde{v}|\sigma(y_1, \ldots, y_n)]
\]
Linear Equilibrium

- The conjectured equilibrium dynamics are

\[
\Delta p_n = \lambda_n y_n + \mu_n q_{n-1}, \\
\Delta q_n = r_n y_n + s_n q_{n-1},
\]

where

\[
\Delta p_n = \lambda_n \left[ y_n - \mathbb{E}[y_n | \sigma(y_1, \ldots, y_{n-1})] \right], \\
q_n = \mathbb{E}[\tilde{a} - \theta_n^R | \sigma(y_1, \ldots, y_n)]
\]

- The optimal orders are linear

\[
\Delta \theta_n^I = \beta_n^I (\tilde{v} - p_{n-1}), \\
\Delta \theta_n^R = \beta_n^R (\tilde{a} - \theta_{n-1}^R) + \alpha_n^R q_{n-1}
\]

- Rebalancer orders are adaptive through \( q_{n-1} \) term
Application of Foster and Viswanathan (1996) approach

- Define linear “hat”-processes

\[
\begin{align*}
\Delta \hat{p}_n &= \lambda_n \hat{y}_n + \mu_n \hat{q}_{n-1} \\
\Delta \hat{q}_n &= r_n \hat{y}_n + s_n \hat{q}_{n-1} \\
\hat{y}_n &= \Delta \hat{\theta}^I_n + \Delta \hat{\theta}^R_n + \Delta w_n
\end{align*}
\]

- Insider’s state-processes are defined by

\[
\begin{align*}
X_n^{(1)} &= \tilde{v} - \hat{p}_n, & X_n^{(2)} &= \hat{q}_n, & X_n^{(3)} &= \hat{\theta}^R_n - \theta^R_n, & X_n^{(4)} &= \hat{q}_n - q_n, & X_n^{(5)} &= \hat{p}_n - p_n,
\end{align*}
\]

which reduces to

\[
\begin{align*}
X_n^{(1)} &= \tilde{v} - p_n, & X_n^{(2)} &= (\hat{\theta}^R_n - \theta^R_n) + (\hat{q}_n - q_n) + \frac{\Sigma_n^{(3)}}{\Sigma_n^{(2)}} (\tilde{v} - \hat{p}_n)
\end{align*}
\]

and the value function is

\[
I_n^{(0)} + \sum_{1 \leq i \leq j \leq 2} I_n^{(i,j)} X_n^{(i)} X_n^{(j)}
\]
Foster and Viswanathan (1996) approach continued

- Rebalancer’s state-processes and value function are defined by

\[
Y_n^{(1)} = \tilde{a} - \hat{\theta}_n^R, \quad Y_n^{(2)} = \hat{q}_n, \quad Y_n^{(3)} = \hat{\theta}_n^R - \theta_n^R, \quad Y_n^{(4)} = \hat{q}_n - q_n, \quad Y_n^{(5)} = \hat{p}_n - p_n,
\]

which reduces to

\[
Y_n^{(1)} = \tilde{a} - \theta_n^R, \quad Y_n^{(2)} = (\hat{p}_n - p_n) + \frac{\sum_n^{(3)}}{\sum_n^{(1)}} (\tilde{a} - \hat{\theta}_n^R - \hat{q}_n), \quad Y_n^{(3)} = q_n
\]

and the value function is

\[
L_n^{(0)} + \sum_{1 \leq i \leq j \leq 3} L_n^{(i,j)} Y_n^{(i)} Y_n^{(j)}
\]
Kalman filter

- Linear projections

\[
\lambda_n = \frac{\beta_n^I \Sigma_n^{(2)} + \beta_n^R \Sigma_n^{(3)}}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2 \beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta},
\]

\[
r_n = \frac{(1 - \beta_n^R)(\beta_n^I \Sigma_n^{(3)} + \beta_n^R \Sigma_n^{(1)})}{(\beta_n^I)^2 \Sigma_n^{(2)} + (\beta_n^R)^2 \Sigma_n^{(1)} + 2 \beta_n^I \beta_n^R \Sigma_n^{(3)} + \sigma_w^2 \Delta},
\]

\[
\mu_n = -(\alpha_n^R + \beta_n^R) \lambda_n,
\]

\[
s_n = -(\alpha_n^R + \beta_n^R) (r_n + 1)
\]

- Conditional moments

\[
\Sigma_n^{(1)} = \mathbb{V} \left[ \tilde{a} - \hat{\theta}_n^R - \hat{q}_n \right] = (1 - \beta_n^R) \left( (1 - \beta_n^R - r_n \beta_n^R) \Sigma_n^{(1)} - r_n \beta_n^I \Sigma_n^{(3)} \right),
\]

\[
\Sigma_n^{(2)} = \mathbb{V} \left[ \tilde{v} - \hat{p}_n \right] = (1 - \lambda_n \beta_n^I) \Sigma_n^{(2)} - \lambda_n \beta_n^R \Sigma_n^{(3)},
\]

\[
\Sigma_n^{(3)} = \mathbb{E} \left[ (\tilde{a} - \hat{\theta}_n^R - \hat{q}_n)(\tilde{v} - \hat{p}_n) \right] = (1 - \beta_n^R) \left( (1 - \lambda_n \beta_n^I) \Sigma_n^{(3)} - \lambda_n \beta_n^R \Sigma_n^{(1)} \right)
\]
Bellman equation restrictions

- **First-order-conditions for individual optimality**

\[
\beta^I_n = \frac{-1 + I_n^{(1,2)} r_n + 2 I_n^{(1,1)} \lambda_n}{2(I_n^{(2,2)} r_n + \lambda_n (-1 + I_n^{(1,2)} r_n + I_n^{(1,1)} \lambda_n))} \\
+ \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(2)}} \left( \frac{-2 I_n^{(2,2)} r_n (-1 + \beta^R_n) + I_n^{(1,2)} \lambda_n - \beta^R_n \lambda_n (I_n^{(1,2)} + 1)}{2(I_n^{(2,2)} r_n + \lambda_n (-1 + I_n^{(1,2)} r_n + I_n^{(1,1)} \lambda_n))} - \beta^R_n \right)
\]

\[
\beta^R_n = \frac{2 L_n^{(1,1)} - L_n^{(1,3)} r_n + \lambda_n + L_n^{(1,2)} \lambda_n}{2 \left( L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n \left( L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)} \lambda_n \right) \right)} \\
+ \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}} \left( \frac{L_n^{(1,2)} - r_n \left( L_n^{(2,3)} + L_n^{(1,3)} \beta^I_n \right) + L_n^{(1,2)} \beta^I_n \lambda_n + 2 \left( L_n^{(1,1)} \beta^I_n + L_n^{(2,2)} \lambda_n \right)}{2 \left( L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n \left( L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)} \lambda_n \right) \right)} - \beta^I_n \right)
\]

\[
\alpha^R_n = \frac{L_n^{(3,3)} \lambda_n - 2 L_n^{(3,3)} r_n - L_n^{(1,3)} ((\alpha_n^R + \beta_n^R)(r_n + 1) - 1) + (\alpha_n^R + \beta_n^R)(2 L_n^{(3,3)} r_n (1 + r_n) + \lambda_n (L_n^{(1,2)} - L_n^{(2,3)} - 2 L_n^{(3,3)} r_n + 2 L_n^{(2,2)} \lambda_n))}{2 \left( L_n^{(3,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n \left( L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)} \lambda_n \right) \right)} \\
- \frac{\Sigma_{n-1}^{(3)}}{\Sigma_{n-1}^{(1)}} \left( \frac{L_n^{(1,2)} - r_n \left( L_n^{(2,3)} + L_n^{(1,3)} \beta^I_n \right) + L_n^{(1,2)} \beta^I_n \lambda_n + 2 \left( L_n^{(1,1)} \beta^I_n + L_n^{(2,2)} \lambda_n \right)}{2 \left( L_n^{(1,1)} - L_n^{(1,3)} r_n + L_n^{(3,3)} r_n^2 + \lambda_n \left( L_n^{(1,2)} - L_n^{(2,3)} r_n + L_n^{(2,2)} \lambda_n \right) \right)} - \beta^I_n \right)
\]

- There are also second-order conditions
Theorem: If constants

\[ \lambda_n, \mu_n, r_n, s_n, \beta^I_n, \beta^R_n, \alpha^R_n, I_n^{(0)}, I_n^{(ij)}, L_n^{(0)}, L_n^{(ij)}, \]

satisfy the restrictions from the Kalman filter and Bellman equations, then a linear Baysian Nash Equilibrium exists with the pricing rule

\[
\Delta p_n = \lambda_n y_n + \mu_n q_{n-1} \\
\Delta q_n = r_n y_n + s_n q_{n-1}
\]

and trading strategies for the insider and rebalancer

\[
\Delta \theta^I_n = \beta^I_n (\tilde{v} - p_{n-1}) \\
\Delta \theta^R_n = \beta^R_n (\tilde{a} - \theta^R_{n-1}) + \alpha^R_n q_{n-1}
\]
Trading motives

- The insider trades since her state variable $\tilde{v} - p_{n-1}$ is informative about
  - Current market mispricing
  - Rebalancer’s trading needs and, thus, about $\Delta \theta^R_n$, and prices:

\[
\mathbb{E}[\Delta p_n | \sigma(\tilde{v}, y_1, \ldots, y_{n-1})] = \lambda_n \left[ \Delta \theta^I_n + \mathbb{E}[\beta^R_n (\tilde{a} - \theta^R_{n-1} - q_{n-1}) | \tilde{v} - p_{n-1}] \right]
\]

- The rebalancer trades given the following considerations:
  - Needs to reach the target $\tilde{a}$
  - Minimizes cost of doing so given the $\lambda_n$ pattern
  - Sunshine trading: Predictable orders have no price impact
  - Private information: $\tilde{a} - \Delta \theta^R_{n-1} - q_{n-1}$ is informative about $\tilde{v} - p_{n-1}$ and, thus, about $\Delta \theta^I_n$ and prices:

\[
\mathbb{E}[\Delta p_n | \sigma(\tilde{a}, y_1, \ldots, y_{n-1})] = \lambda_n \left[ \beta^R_n (\tilde{a} - \theta^R_{n-1} - q_{n-1}) + \mathbb{E}[\beta^I_n (\tilde{v} - p_{n-1}) | \tilde{a} - \theta^R_{n-1} - q_{n-1}] \right]
\]
Numerical analysis

- Like most discrete-time multi-period Kyle models, we do not have analytic comparative statics.
- Numerical experiments illustrate what can happen in the model.
- The numerical algorithm requires us to solve two coupled 5th degree polynomial equations.
Plot of $\lambda_n$ trajectories

A: Kyle (---), $\sigma^2_v = 0.48$ (---), $\sigma^2_a = 1$ (---), $\sigma^2_w = 3.7$ (---).

B: Kyle (---), $\rho = 0$ (---), $\rho = 0.25$ (---), $\rho = 0.47$ (---).

- Blue line represents Kyle’s model
- S-shaped trajectories
- $\lambda_n$ is non-monotone in the amount of informed trading

$\sigma^2_v = 1$, $\sigma^2_w = 4$, $N = 10$, $\rho = 0$ (left only) and $\sigma^2_a = 1$ (right only)
Plot of $\nabla [\tilde{v} - p_{n-1}]$ trajectories

A: Kyle (———), $\sigma^2_{\tilde{a}} = 0.48$ (—), $\sigma^2_v = 1$ (―), $\sigma^2_w = 3.7$ (—.——).

B: Kyle (———), $\rho = 0$ (—), $\rho = 0.25$ (―), $\rho = 0.47$ (—.——).

- Blue line represents Kyle’s model
- When $\rho > 0$, more information revelation

$\sigma^2_v = 1$, $\sigma^2_w = 4$, $N = 10$, $\rho = 0$ (left only) and $\sigma^2_{\tilde{a}} = 1$ (right only)
Plot of $\beta_n^I$ trajectories

A: Kyle (---), $\sigma^2_{\tilde{v}} = 0.48$ (---),
$\sigma^2_{\tilde{a}} = 1$ (---), $\sigma^2_{\tilde{a}} = 3.7$ (---).

B: Kyle (---), $\rho = 0$ (---),
$\rho = 0.25$ (---), $\rho = 0.47$ (---).

- Blue line represents Kyle’s model
- Slightly more aggressive trading
  - Lower price impacts at later dates
  - Holden-Subrahmanyam information competition, but no rat race

$\sigma^2_v = 1$, $\sigma^2_w = 4$, $N = 10$, $\rho = 0$ (left only) and $\sigma^2_{\tilde{a}} = 1$ (right only)
Plot of $\mathbb{E} [\Delta \theta_n^I | \sigma(\tilde{v})]$ given realization $\tilde{v} = 1$

- Blue line represents Kyle’s model
- Note that $\mathbb{E} [\Delta \theta_n^I | \sigma(\tilde{v})]$ is linear in $\tilde{v}$
- $U$-shape, but not big
- Again, no HS rat race

$\sigma_v^2 = 1$, $\sigma_w^2 = 4$, $N = 10$, $\rho = 0$ (left only) and $\sigma_a^2 = 1$ (right only)
Plot of strategy coefficients \((\alpha^R, \beta^R)\)

A: \(\sigma^2_{\tilde{a}} = 0.48\) (---), \(\sigma^2_{\tilde{a}} = 1\) (----), \(\sigma^2_{\tilde{a}} = 3.7\) (-----).

B: \(\rho = 0\) (---), \(\rho = 0.25\) (----), \(\rho = 0.47\) (-----).

- \(\alpha^R_n\)s (below the \(x\)-axis) and \(\beta^R_n\)s (above the \(x\)-axis)
- Decomposition: \(\Delta \theta^R_n = (\alpha^R_n + \beta^R_n)q_{n-1} + \beta^R_n(\tilde{a} - \theta^R_{n-1} - q_{n-1})\)
- Sunshine trading vs order-splitting

\(\sigma^2_v = 1, \sigma^2_w = 4, N = 10, \rho = 0\) (left only) and \(\sigma^2_{\tilde{a}} = 1\) (right only)
Plots of $E[\Delta \theta^R_n | \sigma(\tilde{a})]$ given target realization $\tilde{a} = 1$

- **Note that** $E[\Delta \theta^R_n | \sigma(\tilde{a})]$ **is linear in** $\tilde{a}$
- **U-shape, but not big**

$\sigma^2_v = 1, \sigma^2_w = 4, N = 10, \rho = 0$ (left only) and $\sigma^2_\tilde{a} = 1$ (right only)
Variability of rebalancer’s orders

- U-shaped pattern in conditional SD of $\Delta \theta^R_n$ given $\tilde{a} = 1$
- Over/under shooting can be optimal
- Different from Degryse, de Jong, and van Kervel (2014)

$\sigma^2_v = 1, \sigma^2_w = 4, N = 10, \rho = 0$, and $\sigma^2_{\tilde{a}} = 1$
Predictable components of rebalancer’s trades

A: \[
\frac{\mathbb{E} \left[ \mathbb{E} \left[ \Delta \theta_n^R \middle| \sigma(y_1, \cdots, y_{n-1}) \right] \middle| \sigma(\tilde{a}) \right]}{\mathbb{E} \left[ \Delta \theta_n^R \middle| \sigma(\tilde{a}) \right]} \]
\[
\sigma_{\tilde{a}}^2 = 0.48 (\cdots), \quad \sigma_{\tilde{a}}^2 = 1 (\cdots), \quad \sigma_{\tilde{a}}^2 = 3.7 (\cdots).
\]

B: \[
\frac{\mathbb{E} \left[ \Delta \theta_n^I \middle| \sigma(\tilde{a}, y_1, \cdots, y_{n-1}) \right]}{\Delta \theta_n^R - \mathbb{E} \left[ \Delta \theta_n^R \middle| \sigma(y_1, \cdots, y_{n-1}) \right]} \]

- Plot A (B) is independent of realization of \(\tilde{a}\) (\(\tilde{a}\) and \(y_1, \cdots, y_{n-1}\))
- Sunshine trading component is not large
- Expected informed order strongly offsets rebalancer’s order towards the end

\[
\sigma_v^2 = 1, \quad \sigma_w^2 = 4, \quad N = 10, \quad \rho = 0
\]
Plots of informed and rebalancer order correlation

A: \( \sigma^2_a = 0.48 \) (---), \( \sigma^2_a = 1 \) (---), \( \sigma^2_a = 3.7 \) (---).

B: \( \rho = 0 \) (---), \( \rho = 0.25 \) (---), \( \rho = 0.47 \) (---).

- When \( \text{corr}(\Delta \theta_n^R, \Delta \theta_n^I) < 0 \): symbiotic relationship
- When \( \text{corr}(\Delta \theta_n^R, \Delta \theta_n^I) > 0 \): information competition

\( \sigma^2_v = 1, \sigma^2_w = 4, N = 10, \rho = 0 \) (left only), and \( \sigma^2_a = 1 \) (right only)
Because of rebalancer: $\mathbb{E}[y_n | \sigma(y_1, \ldots, y_{n-1})] = (\alpha_n^R + \beta_n^R)q_{n-1}$

$U$-shaped pattern, but not big

$\sigma_v^2 = 1, \sigma_w^2 = 4, N = 10, \rho = 0$ (left only) and $\sigma_a^2 = 1$ (right only)
Standard deviation of the price changes

A: Kyle (---), $\sigma^2_{a} = 0.48$ (---),
$\sigma^2_{a} = 1$ (---), $\sigma^2_{a} = 3.7$ (---).

B: Kyle (---), $\rho = 0$ (---),
$\rho = 0.25$ (---), $\rho = 0.47$ (---).

- U-shaped pattern in SD of $\Delta p_n$ (unconditional)

$\sigma^2_v = 1$, $\sigma^2_w = 4$, $N = 10$, $\rho = 0$ (left only) and $\sigma^2_{a} = 1$ (right only)
Conclusions

- Material effects from rebalancing on price and order flow dynamics in equilibrium
- Learning by large rebalancers through the trading process
- Strategic complementarities and competition
- Sunshine trading is present, but not necessarily large.
- Non-manipulative trading reversals in rebalancer orders
Future work

- Extension to multiple insiders and rebalancers
- Extension to continuous-time model
- Extension to exponential utility
Independence of $\tilde{a}$

Since the random variables $\Delta \theta^R_n, y_1, \cdots, y_{n-1}, \tilde{a}$ are jointly Gaussian, both $E \left[ E[\Delta \theta^R_n | \sigma(y_1, \cdots, y_{n-1})] | \sigma(\tilde{a}) \right]$ and $E \left[ \Delta \theta^R_n | \sigma(\tilde{a}) \right]$ are linear in $\tilde{a}$. Therefore, the ratio $\frac{E \left[ E[\Delta \theta^R_n | \sigma(y_1, \cdots, y_{n-1})] | \sigma(\tilde{a}) \right]}{E \left[ \Delta \theta^R_n | \sigma(\tilde{a}) \right]}$ is independent of $\tilde{a}$.

$E[\Delta \theta^I_n | \sigma(\tilde{a}, y_1, \cdots, y_{n-1})] = \beta^I_n \frac{\sum_{n}^{(3)}}{\sum_{n}^{(1)}} (\tilde{a} - \theta^R_{n-1} - q_{n-1})$

$\Delta \theta^R_n - E[\Delta \theta^R_n | \sigma(y_1, \cdots, y_{n-1})] = \beta^R_n (\tilde{a} - \theta^R_{n-1} - q_{n-1})$

Therefore, $\frac{E[\Delta \theta^I_n | \sigma(\tilde{a}, y_1, \cdots, y_{n-1})]}{\Delta \theta^R_n - E[\Delta \theta^R_n | \sigma(y_1, \cdots, y_{n-1})]} = \frac{\beta^I_n \sum_{n}^{(3)}}{\beta^R_n \sum_{n}^{(1)}}$. 

31/31