A No-Arbitrage Model Of Liquidity In Financial Markets Involving Stochastic Strings: Applications To High-Frequency Data

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Outline

1. Trading Limit Orders
2. Option Pricing in an Illiquid Market
3. Calibration and Simulation
4. Option Pricing Empirical Analysis
Market vs Limit Orders

A (buy) market order specifies
- how many shares a trader wants to buy,
- that he is willing to buy them at any price.

A (buy) limit order specifies
- how many shares a trader wants to buy,
- at what maximum price he is willing to buy them.
Limit order matching mechanism

Figure: Limit order matching mechanism.
Limit order matching mechanism

**Figure**: Limit order matching mechanism.
Limit order matching mechanism

Figure: Limit order matching mechanism.
Limit order matching mechanism

Figure: Limit order matching mechanism.
Demand v.s. Supply

The order books contain all the information about demand and supply.
The dynamics of Limit Orders in 3D

Figure: Buy Limit Orders of ORCL on April 4, 2011.

Figure: Sell Limit Orders of ORCL on April 4, 2011.
The dynamics of the Clearing Price process

Figure: The dynamics of Oracle Corporation's Clearing Prices on April 1, 2011.
Literature Review: Liquidity Models

Market Manipulation (feedback) Models
- Jarrow (1994),
- Platen and Schweizer (1998),
- Sircar and Papanicolaou (1998),
- Frey (1998),
- Schonbucher and Wilmott (2000),
- Bank and Baum (2004).

Price-taking (competitive) Models
- Cetin, Jarrow, and Protter (2004),
- Cetin and Rogers (2006),
- Cetin, Soner, and Touzi (2009),
- Kallsen and Rheinlaender (2009),
- Gokay and Soner (2011).
Net Demand Curve and Clearing Price

**Definition**
The net demand curve $Q$ is a function $[0, P] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$, which value $Q(p, t, \omega)$ is equal to the difference between the quantity of shares available for purchase and the quantity of shares available for sale at price $p$ at time $t$. For each $p$ the stochastic process $Q(., t, \cdot)$ is a $\mathcal{F}_t$ adapted semimartingale.

**Remark:** For the clearing price to be a diffusion, the demand must be defined on a continuum of limit prices.

**Remark:** The net demand curve should be decreasing in $p$. The easiest way to do that is to model positive processes:
- $Q(0, t)$: total number of buy orders
- $q(p, t)$: density of buy orders + density of sell orders

\[
Q(p, t) = Q(0, t) - \int_0^p q(y, t)dy
\]

**Definition**
The clearing price $\pi(t)$ is a $\mathcal{F}_t$-adapted stochastic process which satisfies market clearing:

\[
Q(\pi(t), t) = 0
\]
The Model

- It can be proved that the optimal strategy of a large trader is to disseminate her orders into infinitesimal orders.
  
  This shows that a continuous demand curve is a plausible model.

- In order to avoid arbitrage, we choose to have as many factors as limit prices

\[ dQ(p, t) = \mu_Q(p, t)dt - \sigma_Q(p, t) \int_{x=0}^{S} b_q(p, x, t) W(ds, dt) \text{ for } 0 < p \leq S \]
Main Result: Market with a Large Trader

**Main Result**

Suppose in addition to our standing assumptions that

C1) The demand curve is decreasing in price and continuous in time;
C2) the volatility $\sigma_Q(p, t)$ is bounded away from zero, uniformly in $p$;
C3) there is no path such that $Q(S, t) \geq 0$ or $Q(0, t) \leq 0$;
C4) The market price of risk equations hold.

Then

F1) there is no arbitrage strategy,
F2) the net demand curve $Q$ is continuous in $t$,
F3) the clearing price $\pi(t)$ is continuous,
F4) The $Q$-measure is also a martingale measure for $\pi(t)$. 
Characterization of the Risk-Neutral Measure $\mathbb{Q}$

**Change of Measure**

$$W^\mathbb{Q}(ds, dt) = W(ds, dt) + \lambda(s, t)dt$$

**Goal:** determine $\lambda$ such that the market price of risk equations hold.
Market Price of Risk Equations

Let \( \{\mu_Q, \sigma_Q, b_Q\} \) be the parameters of net demand. Let the volatility of the clearing price be:

\[
\sigma_{\pi}(t)b_{\pi}(s, t) = -\frac{\sigma_Q(\pi(t), t) \int_{s=0}^{S} b_Q(\pi(t), s, t) ds}{q(\pi(t), t)}
\]

Define

\[
C(\pi, t) = \sigma_{\pi}(t) \int_{s=0}^{S} \frac{\partial[\sigma_Q(p, t, \omega)b_Q(p, s, t)]}{\partial p} \bigg|_{p=\pi} b_{\pi}(s, t) ds
\]

\[
B(\pi, t) = \mu_Q(\pi, t) - \frac{1}{2} \frac{\partial^2 Q(p, t)}{\partial p^2} \bigg|_{p=\pi} + C(\pi, t)
\]

\[
\Sigma(\pi, s, t) = \sigma_Q(\pi, t)b_Q(\pi, s, t)
\]

The market price of risk equations are:

\[
\int_{s=0}^{S} \Sigma(\pi, s)\lambda(s) ds = B(\pi) \quad 0 \leq \pi \leq P
\]
From a "MetaModel" to a Model

Reminder:

\[ q(p, t) \, dp = \text{sum of buy and sell order quantities with limit price in } [p, p + dp] \text{ arriving in } [0, t] \]

Plausible dynamics for \( q \):

- positive process
  - not necessarily increasing: orders can be cancelled
- mean-reverting process
- to be implemented on a computer: \( p \) and \( t \) must take discrete values

Our choice: the exponential of a (vector) Ornstein-Uhlenbeck process.
Model Framework

We define

\[
 dh(p, t, \omega) = \left[ -a_h(p) h(p, t, \omega) + \mu_h(p) \right] dt + \int_{s=0}^{S} \sigma_h(p, s) W(ds, dt, \omega) \\
 q(p, t, \omega) = \exp \left( \int_{x=0}^{p} h(x, y, \omega) dx \right) \quad \text{(Density)} \\
 dQ(p, t, \omega) = \int_{x=0}^{S} q(x, t, \omega) dx \eta(t, \omega) - \int_{x=0}^{p} q(x, t, \omega) dx \quad \text{(Net Demand)} \\
 d\eta(t, \omega) = a_\eta(\bar{\eta} - \eta(t, \omega)) dt \\
 + \sqrt{\eta(t, \omega)(1 - \eta(t, \omega))} \int_{s=0}^{S} \sigma_\eta(s) W(ds, dt, \omega) \quad \text{(Demand Volume Ratio)}
\]
Reasoning of Modeling Framework

- The reason of modeling demand but not demand/supply separately
  Smooth semi-martingale assumption holds for net demand curve

- The reason of modeling the ratio of demand and volume $\eta$ process
  To guarantee the robustness of the estimation: $Q(0) > 0$ and $Q(S) < 0$

- The reason for $Q$ to be twice differentiable
  Need to apply Ito-Wentzell formula.

- The reason for $q$ to be positive
  Guarantee that the net demand is downward sloping (decreases with price).

- The reason for using Stochastic strings
  Rule out arbitrage opportunity
### NYSE Arcabook Data

<table>
<thead>
<tr>
<th>Industry</th>
<th>Exchange</th>
<th>Ticker</th>
<th>Firm</th>
</tr>
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<tbody>
<tr>
<td>Energy</td>
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<td>CVX</td>
<td>Chevron Corporation</td>
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<td>NYSE</td>
<td>XOM</td>
<td>Exxon Mobil Corporation</td>
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<td>Financial Banks</td>
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<td>JPM</td>
<td>JPMorgan Chase &amp; Co.</td>
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<td>NYSE</td>
<td>WFC</td>
<td>Wells Fargo &amp; Company</td>
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<td>Materials and Mining</td>
<td>NYSE</td>
<td>ABX</td>
<td>Barrick Gold Corporation</td>
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<td></td>
<td>NYSE</td>
<td>FCX</td>
<td>Freeport-McMoRan Copper &amp; Gold Inc.</td>
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<td>CSCO</td>
<td>Cisco Systems, Inc.</td>
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<td></td>
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<td>MSFT</td>
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<tr>
<td></td>
<td>NASDAQ</td>
<td>ORCL</td>
<td>Oracle Corporation</td>
</tr>
</tbody>
</table>

**Table:** NYSE Arcabook data selection.
Simulation Process

Market Calibration

- Estimate excess demand $\bar{Q}(p, t)$ from $p$ and $t$ from the high frequency data;
- Calculate the $\eta(t)$ process by $\frac{Q(0, t)}{Q(0, t) - Q(S, t)}$;
- Calculate process $q$ and $h$. Estimate the variance-covariance matrix of $h$ and $\eta$;
- Apply Cholesky decomposition to the correlation matrix, where $\bar{R}dt = Corr\left[\frac{dh_i}{h_i}, \frac{dh_j}{h_j}\right]$
Simulation Process

Excess Demand Simulation
(The process of the simulation is to demonstrate that the calibrated parameters in framework are market consistent)

- Randomly generate $n$ paths for $h$;
- Use $h$ to integrate for $q$ process;
- Simulate $\eta$ process using the correlation estimated in market calibration process;
- Combine $h$, $q$ and $\eta$ to simulate excess demand $Q$. 
Simulation Result

GE stock net demand curve with 04/01/2015 market data calibrated:

Figure: Simulated net demand Q for GE using 04/01/2015 order book calibration.
Characterization of the Risk-Neutral Measure $\mathbb{Q}$

Change of Measure

$$W^Q(ds, dt) = W(ds, dt) + \lambda(s, t)dt$$

**Goal:** determine $\lambda$ such that the market price of risk equations hold.
Market Price of Risk Equations

Let \( \{\mu_Q, \sigma_Q, b_Q\} \) be the parameters of net demand. Let the volatility of the clearing price be:

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\]

\[
\Sigma(\pi, s, t) = \sigma_Q(\pi, t)b_Q(\pi, s, t)
\]

The market price of risk equations are:

\[
\int_{s=0}^{S} \Sigma(\pi, s)\lambda(s)ds = B(\pi) \quad 0 \leq \pi \leq P
\]
Volatility Smile (ORCL)

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<tr>
<th>Strike Price</th>
<th>Implied Volatility</th>
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<td>33.77</td>
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<td>33.80</td>
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<td>33.83</td>
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<td>34.41</td>
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Figure: Simulated vs. real-time volatility smile of ORCL on April 4, 2011. Source: Bloomberg
## Volatility Smile (ABX)

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<td>51.95</td>
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*Figure: Simulated vs. real-time volatility smile of ABX on April 4, 2011. Source: Bloomberg*
Volatility Smile (CSCO)

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<td>16.95</td>
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<td>17.00</td>
<td>4.63%</td>
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<tr>
<td><strong>17.02</strong></td>
<td>4.16%</td>
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<tr>
<td>17.04</td>
<td>4.84%</td>
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<td>17.14</td>
<td>5.85%</td>
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</table>

Figure: Simulated v.s. real-time volatility smile of CSCO on April 4, 2011. Source: Bloomberg
Conclusions

1. We developed a liquidity model with stands between
   - traditional no-arbitrage (option pricing) models and
   - financial economics models.
2. This model uses Ito-Wentzell’s formula and Girsanov’s theorem for Brownian sheets.
3. We give conditions for no-arbitrage, which allows us to price options
4. We specified a model
   - with positive demand density,
   - with mean-reversion,
   - where parameters are centered on the clearing price.
5. We calibrate the model parameters from high frequency stock data of GE as of 04-01-2015. The simulated net demand curve shows a downward sloping shape, as expected.
6. The model generates an implied volatility smile which matches the observed smile much better than the traditional Black-Scholes model.