Location Multiplicative Error Model: Asymptotic Inference and Empirical Analysis

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Outline of Topics

Introduction
  GARCH
  Multiplicative Error Model (MEM)

Location MEM at Order of (p,q)
  Model Specification and Assumption
  An Alternative Representation
  Estimator and Asymptotic Results
  Connection Between MEM and Location MEM

Simulation and Empirical Analysis
  Simulation Study
  Empirical Analysis on IBM Volume per Trade

Conclusion
GARCH and ACD

The ARCH model, proposed by Engle (1982) and later generalized as GARCH by Bollerslev (1986), describes a process (such as the price volatility) \( \{y_t\}_{t=1}^{n} \) as

\[
y_t = \sigma_t \epsilon_t \\
\sigma_t^2 = \omega_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{i=j}^{q} \beta_j \sigma_{t-j}^2
\]

(1)

where \( \epsilon_t \)'s are i.i.d., \( E\epsilon_t = 0 \) and \( E\epsilon_t^2 = 1 \).

The Autoregressive Conditional Duration (ACD) model proposed in Engle and Russell (1998) is one of the extensions of GARCH, specified as

\[
d_t = h_t z_t \\
h_t = \omega_0 + \sum_{i=1}^{p} \alpha_i d_{t-i} + \sum_{i=j}^{q} \beta_j h_{t-j}
\]

(2)

where \( z_t \)'s are i.i.d. and \( z_t > 0, Ez_t = 1 \).
Engle (2002) generalizes ACD as **Multiplicative Error Model (MEM)**.

**Model Specification**

\[
\begin{align*}
    r_t &= h_t z_t \\
    h_t &= \omega + \sum_{i=1}^{p} \alpha_i r_{t-i} + \sum_{i=j}^{q} \beta_j \eta_{t-j} + \gamma' w_t
\end{align*}
\]  

(3)

where \( z_t \)'s are i.i.d., \( z_t \geq 0 \), \( E z_t = 1 \) and \( w_t \) is a vector of weakly exogenous variables.

**Remark:** It applies to a wide range of non-negative financial variables:
- Durations (for example, price duration and volume duration)
- Volume of shares per trade
- Daily high-low range of transaction price
- Ask-bid spread
Motivation For the Extended Model I

IBM stock transaction data on April 8th-12th, 2014

- Intraday trend must be removed before modeling by a smoothing technique.
- There is no typical intrady trend observed in volume per trade.
- The volume per trade data has a lower bound determined by the minimum order size allowed by the exchange.
Motivation For the Extended Model II

Table: Adjusted Duration and Raw Volume

<table>
<thead>
<tr>
<th></th>
<th>1st Quartile/Min</th>
<th>Median/Min</th>
<th>3rd Quartile/Min</th>
<th>Max/Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Duration</td>
<td>8.410256</td>
<td>64.61538</td>
<td>49.20379</td>
<td>1015.385</td>
</tr>
<tr>
<td>R. Volume</td>
<td>1</td>
<td>2.867405</td>
<td>3</td>
<td>81.83</td>
</tr>
</tbody>
</table>

Note: The minimum of volume is not trivial, compared to the scale of the whole sample.

Questions:
- Is there a constant lower bound for volume?
- How to incorporate this lower bound in MEM?
- Can we use the minimum statistic to estimate the lower bound?
- How to estimate the autoregressive equation in MEM with the lower bound?
Location MEM(p,q) and the Matrix Expression

Location Linear MEM at order (p,q):

\[ r_t = \mu_0 + h_t z_t \]  \hspace{1cm} (4)

\[ h_t = \omega + \sum_{i=1}^{p} \alpha_i (r_{t-i} - \mu_0) + \sum_{j=1}^{q} \beta_j h_{t-j} \]  \hspace{1cm} (5)

By Berkes et al. (2003), Location MEM(p,q) can be rewritten as

\[ X_{t+1} = C_t X_t + D \]  \hspace{1cm} (6)

where \( X_t = (h_t, \ldots, h_{t-q-1}, x_{t-1}, \ldots, z_{t-p-1})' \in \mathbb{R}^{p+q-1} \) and \( x_t = r_t - \mu_0 \)

\[ C_t = \begin{bmatrix} \tau_t & \beta_q & \alpha & \alpha_p \\ l_{q-1} & 0 & 0 & 0 \\ \xi_t & 0 & 0 & 0 \\ 0 & 0 & l_{p-2} & 0 \end{bmatrix}, \]

\[ \alpha = (\alpha_2, \ldots, \alpha_{p-1}) \in \mathbb{R}^{p-2}, \tau_t = (\beta_1 + \alpha_1 z_t, \beta_2, \ldots, \beta_{q-1}) \in \mathbb{R}^{q-1}, \]

\[ \xi_t = (z_t, 0, \ldots, 0) \in \mathbb{R}^{q-1}, D = (\omega, 0, \ldots, 0)' \in \mathbb{R}^{p+q-1}, \]
Assumption Based on The Matrix Expression

Assumption 1:
(1) $z_t$’s are i.i.d. and nondegenerate.
(2) $E(\ln \|C_0\|) < \infty$.
(3) $\gamma_L < 0$.
(4) $E|z_t|^s < \infty$ for some $s \geq 1$.

where $\|.\|_d$ is the Euclidean norm in $\mathbb{R}^d$

$$\|M\| = \sup\{\|Mx\|_d/\|x\|_d : x \in \mathbb{R}^d, x \neq 0\},$$

and $\gamma_L$ is defined as

$$\gamma_L = \inf_{0 \leq n \leq \infty} \frac{1}{n+1} E \ln \|C_0C_1 \cdots C_n\|,$$

Proposition 1: Under Assumption 1,
(1) $X_{t+1} = D + \sum_{0 \leq k < \infty} C_t \cdots C_{t-k} D$
(2) $r_t$ and $h_t$ are strictly stationary, and $E|h_t|^s < \infty$, $E|x_t|^s < \infty$. 
Alternative Representation of $h_t$

$h_t$ can be expressed by unique $c_j$'s as

$$h_t = c_0 + \sum_{1 \leq i \leq \infty} c_i (r_{t-i} - \mu_0) \tag{7}$$

where

$$\alpha_1 = c_1, \quad \alpha_2 = c_2 - \beta_1 c_1, \quad \alpha_3 = c_3 - \beta_1 c_2 - \beta_2 c_1, \cdots.$$  

**Remark:** $c_0$, $c_1$, $c_2$, $\cdots$ are functions of $\theta_0 = (\omega, \alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q)$.

- **Definition 5:** $u = (x, s_1, \ldots, s_p, t_1, \ldots, t_q)$ is the vector of unknown parameters for $\theta_0$.
- **Definition 6:** $c_i(u)$'s are the unknown parameters for $c_0, c_1, c_2, \cdots$. 

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Location Multiplicative Error Model
The quasi likelihood function for Location MEM(p,q) takes the form

\[ L^*_n(\mu, u) = \frac{1}{n} \sum_{t=1}^{n} l^*_t(\mu, u) \quad l^*_t(\mu, u) = -\left( \ln h^*_t + \frac{r_t - \mu}{h^*_t} \right) \]

\[ h^*_t = h^*_t(\mu, u) = x + \sum_{i=1}^{p} s_i(r_{t-i} - \mu) + \sum_{j=1}^{q} t_j h^*_{t-j} \]

\[ = c_0(u) + \sum_{i=1}^{t-1} c_i(u)(r_{t-i} - \mu) \]

**Definition 1:** Let \( r_{n(1)} \) be the minimum of \( \{r_t\}_{t=1}^{n} \).

**Definition 2:** Modified QMLE for \( \theta_0 \): \( \hat{\theta} = \arg \max_{u \in \Theta} L^*_n(r_{n(1)}, u) \).
Lemma: Under Assumption 1, $r_{n(1)} \rightarrow_p \mu_0$.

**Proof:** For any $x > 0$,

$$P(r_{n(1)} - \mu_0 > x) = P(h_t z_t > x \text{ for } 1 \leq t \leq n) = P(z_t > \frac{x}{h_t} \text{ for } 1 \leq t \leq n)$$  \hspace{1cm} (9)

For a fixed integer $N$, if $z_t \geq C$ for $N < t \leq n$,

$$h_{n+1} \geq \omega_0 (1 - \beta_0) + (\alpha_0 C + \beta_0) h_n$$
$$\geq (\alpha_0 C + \beta_0)^{n+1-N} h_{0N}$$
$$\geq \omega_0 (\alpha_0 C + \beta_0)^{n+1-N}$$  \hspace{1cm} (10)

Let $A_n$ be the event: \{there exist $N \text{ s.t. } z_t \geq C \text{ for } N < t \leq n$\} and $B = \alpha_0 C + \beta_0$. Then

$$P(A_n) \leq P\left(h_{0n+1} \geq \omega_0 B^{n+1-N}\right)$$
$$\leq P\left(h_{n+1}^\delta \geq \omega_0^\delta B^{(n+1-N)\delta}\right)$$
$$\leq \frac{Eh_{0n+1}^\delta}{\omega_0^\delta B^{(n+1-N)\delta}}$$  \hspace{1cm} (11)
Consistency of The Minimum Statistic II

For some $\delta$ and $C$, we have $\lim_{n \to \infty} P(A_n) = 0$. Since $A_n \downarrow A$, where

$$A = \{ \text{there exist } N \text{ s.t. } z_t \geq C \text{ for all } t > N \},$$

$$P(A^c) = 1 - \lim_{n \to \infty} P(A_n) = 1.$$  

That is, for any integer $N$, there exist $t_N > N$ s.t. $z_{t_N} < C$. Select a monotone increasing sequence $\{t_N\}_{N=1}^\infty$ such that $z_{t_N} < C$, obviously $t_N \to \infty$ as $N \to \infty$.

Let $n = t_N$, then by the assumption that $z_t$'s are independent

$$P \left( z_t > \frac{x}{h_t} \text{ for } 1 \leq t \leq n \right) \leq P \left( z_{t_1} > \frac{x}{h_{t_1}}, \ldots, z_{t_N} > \frac{x}{h_{t_N}} \right) \leq P \left( z_{t_1} < C, \ldots, z_{t_N} < C \right) \leq P(z_t < C)^N \quad (12)$$

The right hand side of the third inequality in equation (12) approaches to 0 as $N \to \infty$. That is $P(r_{n(1)} - \mu_0 > x) \to 0$ as $n \to \infty$. 

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Location Multiplicative Error Model
Assumption 2: \( E|r_{n(1)} - \mu_0|^s = o\left(\frac{1}{\sqrt{n}}\right) \) for some \( s < 1 \).

Theorem: Under Assumption 1 and 2 and if \( E \ln h_t < \infty \) holds,

1. \( \hat{\theta} \to_p \theta_0 \).

2. \( \sqrt{n}(\hat{\theta} - \theta_0) \to_D N(0, V_0) \) where \( V_0 = G_0^{-1}C_0G_0^{-1} \), \( C_0 = -E(\nabla u_l(\theta_0)\nabla u_l(\theta_0)') \) and \( G_0 = -E\nabla^2 u_l(\theta_0) \).

3. \( \hat{V}_n = \hat{G}_n^{-1}(\hat{\theta})\hat{C}_n(\hat{\theta}_n)\hat{G}_n^{-1}(\hat{\theta}) \to_p V_0 = G_0^{-1}C_0G_0^{-1} \), where

\[
\hat{G}_n(\theta) = -\frac{1}{n} \sum_{t=1}^{n} \nabla^2 \theta l^*_t(r_{n(1)}, \theta), \quad \hat{C}_n(\theta) = \frac{1}{n} \sum_{t=1}^{n} \nabla_\theta l^*_t(r_{n(1)}, \theta) \nabla_{\theta'} l^*_t(r_{n(1)}, \theta).
\]
Suppose $r_t$ is a Location MEM(p,q) process.

Let $H_t = \mu_0 + h_t$ and $x_t = \frac{\mu_0 + h_t z_t}{\mu_0 + h_t}$, then

$$
\begin{align*}
  r_t &= H_t x_t \\
  H_t &= \omega' + \sum_{i=1}^{p} \alpha_i r_{t-i} + \sum_{j=1}^{q} \beta_j H_{t-j}
\end{align*}
$$

(13)

where $\omega' = \omega + (1 - \sum_{i=1}^{p} \alpha_i - \sum_{i=1}^{q} \beta_j) \mu_0$ and $E(x_t) = 1$.

**Remark:**

- $x_t$'s are not i.i.d..
- When $\mu_0$ is trivial compared to $h_t$, $x_t$ can be approximated by $z_t$.
- When $\mu_0$ is significant, the estimation for process (13) is not valid.
Consistency and Asymptotic Normality for Location MEM(1,1)

<table>
<thead>
<tr>
<th></th>
<th>ω</th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>True Value</strong></td>
<td>0.03</td>
<td>0.1</td>
<td>0.85</td>
</tr>
<tr>
<td><strong>Exponential</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.030890</td>
<td>0.100333</td>
<td>0.84802</td>
</tr>
<tr>
<td>S.D.</td>
<td>0.005573</td>
<td>0.009727</td>
<td>0.016241</td>
</tr>
<tr>
<td><strong>Weibull</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.030661</td>
<td>0.101692</td>
<td>0.846497</td>
</tr>
<tr>
<td>S.D.</td>
<td>0.005223</td>
<td>0.008585</td>
<td>0.014517</td>
</tr>
<tr>
<td><strong>Gamma</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>0.029305</td>
<td>0.102111</td>
<td>0.845881</td>
</tr>
<tr>
<td>S.D.</td>
<td>0.0052</td>
<td>0.008652</td>
<td>0.015456</td>
</tr>
</tbody>
</table>

Table: Modified QMLE of Location MEM(1,1) for Different Distributions.
QQ Plot of standardized estimates for different distributions

- Exponential
- Weibull
- Gamma
- Standard Normal

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Location Multiplicative Error Model
Estimates for DGP of Location MEM(2,2) with $\mu_0=8$

<table>
<thead>
<tr>
<th></th>
<th>$\omega$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>True Value in L-MEM</td>
<td>0.03</td>
<td>0.05</td>
<td>0.1</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>True Value in MEM</td>
<td>0.08</td>
<td>0.05</td>
<td>0.1</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>Est. of L-MEM</td>
<td>0.03</td>
<td>0.05</td>
<td>0.1</td>
<td>0.503</td>
<td>0.296</td>
</tr>
<tr>
<td>(S.D.)</td>
<td>(0.003)</td>
<td>(0.006)</td>
<td>(0.009)</td>
<td>(0.084)</td>
<td>(0.074)</td>
</tr>
<tr>
<td>Est. of MEM</td>
<td>0.0055</td>
<td>0.06</td>
<td>0.1</td>
<td>0.408</td>
<td>0.433</td>
</tr>
<tr>
<td>(S.D.)</td>
<td>(0.0001)</td>
<td>(0.006)</td>
<td>(0.008)</td>
<td>(0.072)</td>
<td>(0.066)</td>
</tr>
</tbody>
</table>
QQ Plot of standardized estimates for $\mu_0 = 8$ at order (2,2)
Empirical Improvement at Lag of (1,2): IBM Trade Volume at NYSE April 8th-12th, 2013.

Table: Improvement on Raw Data

<table>
<thead>
<tr>
<th>Location MEM</th>
<th></th>
<th></th>
<th></th>
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<tbody>
<tr>
<td>Est. of $\theta_0$</td>
<td>2.055</td>
<td>0.029</td>
<td>0.44</td>
<td>0.51</td>
</tr>
<tr>
<td>S.D.</td>
<td>0.68</td>
<td>0.005</td>
<td>0.125</td>
<td>0.125</td>
</tr>
<tr>
<td>LLH</td>
<td>-56772.35</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ljung-Box Q(15)</td>
<td>23.6975</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>p Value: Q(15)</td>
<td>0.07041</td>
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<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>MEM</th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Est. of $\theta_0$</td>
<td>4.818</td>
<td>0.033</td>
<td>0.414</td>
<td>0.529</td>
</tr>
<tr>
<td>S.D.</td>
<td>1.675</td>
<td>0.005</td>
<td>0.151</td>
<td>0.153</td>
</tr>
<tr>
<td>LLH</td>
<td>-62598.45</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ljung-Box Q(15)</td>
<td>32.2416</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p Value: Q(15)</td>
<td>0.005968</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

–Main Results:
- Asymptotic properties of the modified QMLE for Location MEM.
- Simulation study and real data analysis illustrate the asymptotic results and improvement of goodness-of-fit.
- There is no need to remove intraday trend while modeling volume per trade.

–Future Work:
- Zero-Augmented MEM as proposed by Hautsch et al. (2014) and asymptotic analysis on a mixture density QMLE.
- Asymptotic Inference on Location MEM allowing for weakly-exogenous variables as GARCH-X model in Han (2013).
References I


References II


Thank you!