Vol, Skew, & Smile Trading

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Nov. 5th, 2016
Introduction

• In stochastic volatility models, the folk wisdom is that the height of the implied vol curve reflects the market’s expectations of future realized volatility, the slope reflects the covariation of this volatility with the underlying, and the curvature reflects the volatility of volatility.

• An investor’s views might differ from the market’s view as expressed through the level, slope, and curvature of the implied volatility curve.

• How can an investor trade when their views differ from the market on either the mean of future volatility, the covariation of future volatility with the underlying, or the volatility of future volatility?
Three Rough Definitions

- A vol trade is a position that gains on average when realized vol is sufficiently high.
- A skew trade is a position that gains on average when the realized covariation of vol with the underlying is sufficiently high.
- A smile trade is a position that gains on average when the vol of vol is sufficiently high.
- When either the level, slope, or curvature of implied vol is below an investor's view on vol, covariation, or vol-vol, exactly how should an investor trade so as to profit on average from being correct?
Motivation

- How can you trade so as to be profitable either over all paths or at least on average when:
  
  1. You know realized vol will definitely exceed 10% and yet ATM implied vol is below 10%.
  2. You know that the correlation of every implied vol with the underlying will realize positive and yet the OTM call implied vol is below an equally OTM put implied vol.
  3. You know that implied vol’s are themselves volatile and yet the average of two equally OTM put and call impliieds is at or below the ATM implied?

- We actually respectively consider vol, skew, or smile trading under a view on the realized variance of the log of the underlying, the realized covariation of the log underlying with log implied vol, or the realized variance of log implied vol. Given a view on one, we assume no clue on the other two.
Financial Setting

• Working in a foreign exchange (FX) context, we assume zero interest rates throughout this talk.

• Let $S_t > 0$ be the underlying spot FX rate at time $t \in [0, T]$ expressed as domestic currency per foreign currency unit.

• Let $I_t(K)$ denote the Black Merton Scholes implied vol at strike rate $K > 0$ for some fixed maturity date $T \geq t \geq 0$.

• We suppose that a market maker continuously quotes the FX level $S_t$ and an entire implied vol curve $I_t(K), K > 0$.

• We will be assuming that the risk neutral dynamics of the FX rate and all co-terminal implied vol’s are driftless geometric Brownian motions, generalized to have arbitrary unknown stochastic volatility.

• We assume no frictions, in particular markets are always open and bid ask spreads vanish.
Continuous Rebalancing

- In our financial setting, we will show that vol trading is achieved by continuously rebalancing a position in an always ATM straddle.

- Skew trading involves continuously rebalancing a long position in an out-of-the-money (OTM) call, while continuously rebalancing a short position in an equally OTM put.

- Finally, smile trading involves continuously rebalancing long positions in an equally OTM call and put, while continuously rebalancing a short position in ATM straddles.

- All three strategies require continuous rebalancing of option holdings, otherwise known as high frequency trading.
Implications of No Arbitrage

• We assume no arbitrage between the two currencies, but we allow a possible arbitrage whenever an option is involved.

• The absence of arbitrage between the two currencies implies the existence of two equivalent martingale measures $Q_-$ and $Q_+$. 

• Under $Q_-$, $S$ is a (non-trivial local) martingale, while under $Q_+$ $1/S$ is a (non-trivial local) martingale.

• Since $S$ is risky, $\ln S$ has negative drift under $Q_-$, but positive drift under $Q_+$. 

The Black Merton Scholes Model

- Recall that under $Q_-$, $S > 0$ is a (non-trivial local) martingale.
- The Black Merton Scholes (BMS) model further restricts $S$ to have continuous paths and to have constant volatility.
- To see what this means, let $Q^{b}_\pm$ be the two equivalent martingale measures $Q_-$ and $Q_+$ in the BMS model.
- Recall we assume zero interest rates over $[0, T]$.
- Under $Q^b_-$, the BMS model assumes that $S$ solves the following stochastic differential equation (SDE):

$$dS_t = \sigma S_t dW_t, \quad t \in [0, T],$$

where $\sigma > 0$ is the constant instantaneous volatility of $S$. Here, $W$ is a $Q_-$ standard Brownian motion.
Let the time to maturity \( \tau \equiv T - t \) be the difference between the maturity date \( T \) and the calendar time \( t \).

For \( \sigma > 0, \tau > 0 \), and 0 int. rates, the BMS put pricing formula is:

\[
P^b(S, \sigma, \tau; K) \equiv KN(z_-(K/S, \sigma \sqrt{\tau})) - SN(z_+(K/S, \sigma \sqrt{\tau})),
\]

where for \( z \in \mathbb{R} \), \( N(z) \equiv \int_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \) is the standard normal cumulative distribution function, whose arguments are defined as:

\[
z_\pm(K/S, \sigma \sqrt{\tau}) \equiv \frac{\ell_\pm(K/S, \sigma \sqrt{\tau})}{\sigma \sqrt{\tau}}, \ell_\pm(K/S, \sigma \sqrt{\tau}) \equiv \ln(K/S) \pm \sigma^2 \tau/2.
\]

\( \ell_\pm(K/S, \sigma \sqrt{\tau}) = \ln(K/S) - E^Q_\pm [\ln(S_T/S_t) | S_t = S] \) is called log moneyness.
BMS Implied Volatilities

• At all times $t \in [0, T]$, our market-maker quotes BMS implied volatility (IV) by strike, $I_t(K)$, $K > 0$ for a fixed maturity date $T$.

• Let $P_t(K)$ be the time $t$ market price of the put with strike rate $K > 0$ and maturity date $T \geq t$.

• By the definition of BMS IV, the time $t$ market price of the put is:

$$P_t(K) = P^b(S_t, I_t(K), T - t; K), \quad K > 0, t \in [0, T].$$

• For each fixed strike $K > 0$, IV is the wrong volatility to put into the wrong put pricing formula $P^b$ to get the right put price $P_t(K)$.

• However, for varying strike $K > 0$, the insertion of some IV curves $I_t(K), K > 0$ into the same wrong put pricing formula $P^b$ leads to a wrong (i.e. arbitrageable) put price curve, $P_t(K), K > 0$. 
Relative Gamma in the BMS Model

- Practitioners call partial derivatives of the option price “Greeks”.
- For example, differentiating the BMS put option pricing formula twice w.r.t. $S$, the put’s gamma is given by:

$$P_{11}^b(S, \sigma, \tau; K) = \frac{N'(z_+(K/S, \sigma\sqrt{\tau}))}{S\sigma\sqrt{\tau}}, \quad S > 0, \sigma > 0, \tau > 0, K > 0.$$ 

- We define the relative gamma of the put as:

$$S^2P_{11}^b(S, \sigma, \tau; K) = S \frac{N'(z_+(K/S, \sigma\sqrt{\tau}))}{\sigma\sqrt{\tau}} = K \frac{N'(z_-(K/S, \sigma\sqrt{\tau}))}{\sigma\sqrt{\tau}}.$$ 

- Relative gamma is measured in the same units as the option premium. All of the relative greeks will have this important property.
Other Greeks in the BMS Model

- We will be interested in all the greeks that arise from an application of Itô’s formula in our setting.

- It turns out that all of the greeks that we are interested in have formulas that are just simple multiples of the formula for relative gamma on the last slide.

- Letting $R\Gamma \equiv S^2 P_{11}^b(S, \sigma, \tau) = \frac{KN'(z_-)}{\sigma \sqrt{\tau}}$ denote relative gamma, the table below lists the greeks that we are interested in:

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
<th>Link to Relative Gamma $R\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Vega</td>
<td>$\sigma P_2^b(S, \sigma, \tau)$</td>
<td>$R\Gamma \sigma^2 \tau$</td>
</tr>
<tr>
<td>Relative Vanna</td>
<td>$S\sigma P_{12}^b(S, \sigma, \tau)$</td>
<td>$R\Gamma \ell_-$</td>
</tr>
<tr>
<td>Relative Volga</td>
<td>$\sigma^2 P_{22}^b(S, \sigma, \tau)$</td>
<td>$R\Gamma \ell_- \ell_+$</td>
</tr>
<tr>
<td>Theta</td>
<td>$-P_3^b(S, \sigma, \tau)$</td>
<td>$-R\Gamma \frac{\sigma^2}{2}$</td>
</tr>
</tbody>
</table>

where $\ell_{\pm}(\frac{K}{S}, \sigma \sqrt{\tau}) \equiv \ln(\frac{K}{S}) \mp \sigma^2 \tau / 2$ is a log moneyness measure.
Dynamical Restrictions on Spot FX Rate

• Until now, we have not imposed any dynamical restrictions. We have merely presented the zero-rates BMS-model European-put pricing formula, defined implied volatility, and calculated several greeks of interest.

• We now restrict the dynamics of the underlying spot FX rate $S$ and the implied volatility curve $I(K), \ K > 0$. Suppose that under $\mathbb{Q}_-$ and zero interest rates, $S$ solves the following SDE:

$$dS_t = \sigma_t S_t dW_t, \quad t \in [0, T],$$

where recall $W$ is a $\mathbb{Q}_-$ standard Brownian motion.

• The subscript $t$ on $\sigma$ indicates that the instantaneous volatility of $S$ is now a stochastic process. We treat $\sigma_0$ as a positive random variable whose $\mathbb{Q}_-$ law is unknown. We do not assume that investors are able to directly specify the $\mathbb{Q}_-$ dynamics of the instantaneous volatility process $\sigma$. 
Dynamical Restrictions on Implied Volatilities

- We assume instead that the IV curve $I_t(K), K > 0$ solves:

  $$dI_t(K) = \omega_t I_t(K) dZ_t, \quad K > 0, t \in [0, T],$$

  where $Z$ is a one-dimensional $\mathbb{Q}$-standard Brownian motion.

- We assume that the initial IV curve $I_0(K), K > 0$ is an observed strictly positive function and that the volvol process $\omega_t \in (0, \infty)$ is an unobserved positive bounded process. As a result, subsequent IV curves $I_t(K), K > 0, t \in (0, T]$ are all strictly positive.

- Since the volvol process $\omega_t$ does not depend on $I$ or $K$, all of the implied volatilities undergo the same proportional shifts.

- Our dynamics imply that the ratio of two IV’s at any future time is just the ratio of the two initial IV’s:

  $$\frac{I_t(K)}{I_t(K')} = \frac{I_0(K)}{I_0(K')}, \quad K \neq K', t \in (0, T].$$
Correlation and Covariation Processes

- Recall our dynamical restrictions on $S$ and $I(K), K > 0$:
  \[ dS_t = \sigma_t S_t dW_t, \quad dl_t(K) = \omega_t l_t(K) dZ_t, \quad K > 0, t \in [0, T], \]
  where $W, Z$ are $Q_-$ standard Brownian motions (SBM’s).
- The volatilities of $S$ and $I(K)$ are themselves strictly positive stochastic processes, whose initial levels $\sigma_0$ and $\omega_0$ and subsequent dynamics $d\sigma_t$ and $d\omega_t$ are unspecified random variables.
- Let $\rho_t \in [-1, 1]$ be the bounded stochastic process governing the correlation between the 2 SBM’s $W$ and $Z$ at time $t \in [0, T]$:
  \[ d\langle W, Z \rangle_t = \rho_t dt. \]
  We assume that $\rho_0$ is a random variable whose law has support in $[-1, 1]$, but whose $Q_-$ law is unknown.
- The covariation process $\gamma_t$ is defined as the coefficient of $dt$ in
  \[ d\langle \ln S, \ln l(K) \rangle_t. \] The SDE’s imply $\gamma_t \equiv \sigma_t \rho_t \omega_t$, which are all unspecified.
At-the-Money Strike Rate, IV, and Straddle

• Recall our log moneyness measure $\ell_-(\frac{K}{S}, \sigma \sqrt{T}) \equiv \ln\left(\frac{K}{S}\right) + \frac{\sigma^2 T}{2}$.

• At any time $t \in [0, T]$, we define the at-the-money (ATM) strike rate as the unique strike rate $K^a_t > 0$ that zeroes out log-moneyness:

$$\ell_\left(\frac{K^a_t}{S_t}, I^a_{at} \sqrt{T - t}\right) = 0,$$

where $I^a_{at} \equiv I^a_t(K^a_t)$ denotes the ATM IV quoted at the varying time $t \geq 0$ for the fixed maturity date $T \geq t$.

• A straddle is a long position in one call and one put with the same underlying, strike, and maturity.

• A straddle maturing at $T > 0$ is said to be ATM at the fixed time $t \in [0, T]$ and fixed spot level $S_t > 0$ if the common strike rate of the put and call is $K^a_t$.

• We allow a trader to continuously trade ATM straddles. We also allow continuous trading in two out-of-the-money (OTM) options.
Out-of-the-Money Put and Call

- For some time-varying strike rate $K^p_t < K^a_t$, let $I_{pt} \equiv I_t(K^p_t)$ be an OTM put IV at time $t \in [0, T]$, defined so that:

  \[ P_t(K^p_t) = P^b(S_t, I_{pt}, T - t; K^p_t), \quad t \in [0, T]. \]

- The BMS call value with fixed strike rate $K > 0$ and fixed maturity date $T > t \geq 0$ is given by:

  \[ C^b(S, \sigma, \tau; K) \equiv SN(-z_-(K/S, \sigma \sqrt{\tau})) - KN(-z_+(K/S, \sigma \sqrt{\tau})), \tau \equiv T - t. \]

- For some time-varying strike rate $K^c_t > K^a_t$, let $I_{ct} \equiv I_t(K^c_t)$ denote the OTM call IV at time $t \in [0, T]$, defined so that:

  \[ C_t(K^c_t) = C^b(S_t, I_{ct}, T - t; K^c_t), \quad t \in [0, T]. \]
Instantaneous Gains

- We define the instantaneous gain \( g_{P_t}(K^P_t) \) on the OTM put by:

\[
g_{P_t}(K^P_t) \equiv \left. \left[ dP^b(S_t, I_t(K), T - t; K) \right] \right|_{K=K^P_t}, \quad t \in [0, T].
\]

- Let \( A^b(S, \sigma, \tau; K) = P^b(S, \sigma, \tau; K) + C^b(S, \sigma, \tau; K) \) be the BMS model value of a straddle and let

\[
A_t(K^a_t) = A^b(S_t, I_t(K^a_t), T - t; K^a_t) = P_t(K^a_t) + C(K^a_t)
\]

be the ATM straddle value at time \( t \in [0, T] \).

- We define the instantaneous gain \( g_{A_t}(K^a_t) \) on the ATM straddle by:

\[
g_{A_t}(K^a_t) \equiv \left. \left[ dA^b(S_t, I_t(K), T - t; K) \right] \right|_{K=K^a_t}, \quad t \in [0, T].
\]

- We also define the instantaneous gain \( g_{C_t}(K^c_t) \) on the OTM call by:

\[
g_{C_t}(K^c_t) \equiv \left. \left[ dC^b(S_t, I_t(K), T - t; K) \right] \right|_{K=K^c_t}, \quad t \in [0, T].
\]
Instantaneous Gain of a 3 Strike Option Portfolio

• Let $V_t$ be the value of the following three strike rate option portfolio at time $t \in [0, T]$:

$$V_t \equiv \eta_t^p P_t(K_t^p) + \eta_t^a A_t(K_t^a) + \eta_t^c C_t(K_t^c), \quad t \in [0, T].$$

• We define the instantaneous gain on this portfolio at $t \in [0, T]$ as:

$$gV_t \equiv \eta_t^p gP_t(K_t^p) + \eta_t^a gA_t(K_t^a) + \eta_t^c gC_t(K_t^c), \quad t \in [0, T].$$

• Thus, the inst. gain $gV_t$ differs from the total derivative $dV_t$ in that the former suppresses the time variation of the option holdings.

• As a result, the instantaneous gain in value of the option portfolio is just a linear combination of the previously defined instantaneous gains in each option price.
Decomposing Inst. Gain of OTM Put

• Recall that the instantaneous gain $g_{P_t}(K^p_t)$ on the OTM put is

$$g_{P_t}(K^p_t) \equiv \left[ dP^b(S_t, I_t(K), T - t; K) \right]_{K=K^p_t} , \quad t \in [0, T],$$

where under $\mathbb{Q}_-$, the spot FX rate $S$ and the IV curve $I(K), K > 0$ solve the SDE’s:

$$dS_t = \sigma_t S_t dW_t, \quad dl_t(K) = \omega_t I_t(K) dZ_t, \quad d\langle W, Z \rangle_t = \rho_t dt.$$

• For $t \in [0, T]$, Itô’s formula implies that the instantaneous gain $g_{P_t}(K^p_t)$ on the OTM put decomposes as:

$$g_{P_t}(K^p_t) = P^b_1(S_t, I_{ pt}, T - t; K^p_t) dS_t + P^b_2(S_t, I_{ pt}, T - t; K^p_t) dl_{ pt} + G^p_t dt,$$

where since $S$ and $l_p$ are $\mathbb{Q}_-$ local martingales, $G^p_t$ is the *mean gain rate* under $\mathbb{Q}_-$ on the OTM put at time $t \in [0, T]$, given on the next slide.
Mean Gain Rate of OTM Put

- Under $\mathbb{Q}_-$, the mean gain rate on the OTM put is:

$$G_t^p = P_{11}^b(S_t, l_{pt}, T - t; K_t^p) \frac{d\langle S \rangle_t}{2dt} + P_{12}^b(S_t, l_{pt}, T - t; K_t^p)) \frac{d\langle S, l_p \rangle_t}{dt} + P_{22}^b(S_t, l_{pt}, T - t; K_t^p)) \frac{d\langle l_p \rangle_t}{2dt} - P_3^b(S_t, l_{pt}, T - t; K_t^p)).$$

- Since $dS_t = \sigma_t S_t dW_t$, $dl_t(K) = \omega_t l_t(K) dZ_t$, and $d\langle W, Z \rangle_t = \rho_t dt$, the second order variations are:

$$\frac{d\langle S \rangle_t}{dt} = \sigma_t^2 S_t^2, \quad \frac{d\langle S, l_p \rangle_t}{dt} = S_t \sigma_t \rho_t \omega_t l_{pt} \equiv S_t \gamma_t l_{pt}, \quad \frac{d\langle l_p \rangle_t}{dt} = \omega_t^2 l_{pt}^2.$$

- Substituting implies that the $\mathbb{Q}_-$ mean gain rate on the OTM put is:

$$G_t^p = S_t^2 P_{11}^b \frac{\sigma_t^2}{2} + S_t l_{pt} P_{12}^b \gamma_t + l_{pt}^2 P_{22}^b \frac{\omega_t^2}{2} - P_3^b,$$

with all greeks evaluated at $(S_t, l_{pt}, T - t; K_t^p)$. 
Mean Gain Rate of OTM Put (Con’d)

• Recall that the $\mathbb{Q}_-$ mean gain rate on the OTM put is:

$$G_t^p = S_t^2 P_{11}^b \frac{\sigma_t^2}{2} + S_t l_{pt} P_{12}^b \gamma_t + I_{pt}^2 P_{22}^b \frac{\omega_t^2}{2} - P_3^b,$$

at $(S_t, l_{pt}, T - t; K_t^p)$.

• Hence, the $\mathbb{Q}_-$ mean gain on the OTM put at time $t \in [0, T]$ is a linear combination of the put’s relative gamma: $R\Gamma_t^p \equiv S_t^2 P_{11}^b$, its relative vanna: $S_t l_{pt} P_{12}^b = R\Gamma_t^p l_{-t}^p, l_{-t}^p \equiv -l_-(K_t^p / S_t, l_{pt} \sqrt{T - t})$, its relative volga: $I_{pt}^2 P_{22}^b = R\Gamma_t^p l_{-t}^p l_{+t}^p, l_{+t}^p \equiv l_+(K_t^p / S_t, l_{pt} \sqrt{T - t})$, and its theta: $-P_3^b = -R\Gamma_t^p \frac{l_{pt}^2}{2}$.

• Thus, the $\mathbb{Q}_-$ mean gain rate on the OTM put simplifies to:

$$G_t^p = R\Gamma_t^p \left[ \frac{\sigma_t^2}{2} + \gamma_t l_{-t}^p + \frac{\omega_t^2}{2} l_{-t}^p l_{+t}^p - \frac{l_{pt}^2}{2} \right].$$
Inst. Gains of OTM Options & ATM Straddle

- The inst. gain on the OTM put at time $t \in [0, T]$ is given by:

$$ g_{P_t}(K_t^p) = P_1^b(S_t, l_{pt}, T - t)\sigma_t S_t dW_t + P_2^b(S_t, l_{pt}, T - t)\omega_t l_{pt} dZ_t + R\Gamma_t^p \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t}^p + \frac{\omega_t^2}{2} \ell_{-t}^p \ell_{+t}^p - \frac{l_{pt}^2}{2} \right] dt. $$

- The inst. gain on the OTM call is analogously given by:

$$ g_{C_t}(K_t^c) = C_1^b(S_t, l_{ct}, T - t)\sigma_t S_t dW_t + C_2^b(S_t, l_{ct}, T - t)\omega_t l_{ct} dZ_t + R\Gamma_t^c \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t}^c + \frac{\omega_t^2}{2} \ell_{-t}^c \ell_{+t}^c - \frac{l_{ct}^2}{2} \right] dt. $$

- The inst. gain on the ATM straddle at time $t \in [0, T]$ is simpler:

$$ g_{A_t}(K_t^a) = A_1^b(S_t, l_{at}, T - t)\sigma_t S_t dW_t + A_2^b(S_t, l_{at}, T - t)\omega_t l_{at} dZ_t + R\Gamma_t^a \left[ \frac{\sigma_t^2}{2} - \frac{l_{at}^2}{2} \right] dt, $$

since vanna and volga vanish.
Inst. Gains of 3 Strike Option Portfolio

The inst. gain on the 3 strike rate option portfolio is:

\[ gV_t \equiv \eta_t^p gP_t(K_t^p) + \eta_t^a gA_t(K_t^a) + \eta_t^c gC_t(K_t^c) \]

\[ = G_t^\gamma dt + \Delta_t^\gamma \sigma_t S_t dW_t + R_{\gamma}^t \omega_t dZ_t, \]

where \( G_t^\gamma \) is the \( \mathbb{Q} \)–mean gain rate on the option portfolio.

\[ G_t^\gamma = \eta_t^p R\Gamma_t^p \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t}^p + \frac{\omega_t^2}{2} \ell_{-t}^p \ell_{+t}^p - \frac{l_{pt}^2}{2} \right] + \eta_t^a R\Gamma_t^a \left[ \frac{\sigma_t^2}{2} - \frac{l_{at}^2}{2} \right] \]

\[ + \eta_t^c R\Gamma_t^c \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t}^c + \frac{\omega_t^2}{2} \ell_{-t}^c \ell_{+t}^c - \frac{l_{ct}^2}{2} \right], \]

\( \Delta_t^\gamma \) is the foreign currency delta of the option portfolio:

\[ \Delta_t^\gamma = \eta_t^p P_{1}^b(S_t, I_{pt}, \tau) + \eta_t^a A_{2}^b(S_t, I_{at}, \tau) + \eta_t^c C_{1}^b(S_t, I_{ct}, \tau), \]

\( \tau \equiv T - t \), while \( R_{\gamma}^t \) is the relative vega of the option portfolio:

\[ R_{\gamma}^t \equiv \eta_t^p I_{pt} P_{2}^b(S_t, I_{pt}, \tau) + \eta_t^a I_{at} A_{2}^b(S_t, I_{at}, \tau) + \eta_t^c I_{ct} C_{2}^b(S_t, I_{ct}, \tau) \]

\[ = (T - t) \left[ \eta_t^p R\Gamma_t^p l_{pt}^2 + \eta_t^a R\Gamma_t^a l_{at}^2 + \eta_t^c R\Gamma_t^c l_{ct}^2 \right]. \]
Recall that the inst. gain on the 3 strike rate option portfolio is:

\[ gV_t = G^v_t dt + \Delta^v_t \sigma_t S_t dW_t + Rv^v_t \omega_t dZ_t, \quad t \in [0, T], \]

where:

\[
G^v_t = \eta^p_t R \Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^p_t + \frac{\omega^2_t}{2} \ell^p_t \ell^p_{-t} - \frac{I^2_{pt}}{2} \right] + \eta^a_t R \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right] + \eta^c_t R \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^c_t + \frac{\omega^2_t}{2} \ell^c_{-t} \ell^c_{+t} - \frac{I^2_{ct}}{2} \right].
\]

Thus, the inst. gain is the sum of the signal \( G^v_t dt \) plus noise.

In general, the signal depends on the inst. variance rate \( \sigma^2_t \) of \( \ln S \), the inst. covariation rate \( \gamma_t \equiv \sigma_t \rho_t \omega_t \) between \( \ln S \) and \( \ln I(K) \), and the inst. variance rate \( \omega^2_t \) of \( \ln I(K) \).

Suppose that a trader can only forecast 1 of these 3 processes. Can a positive signal be generated w/o knowing the other 2 processes?
Vol Trade

• Recall that the \( \mathbb{Q}_- \) mean gain rate is:

\[
G_t^\mathbb{Q}_- = \eta_t^p R\Gamma_t^p \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t}^p + \frac{\omega_t^2}{2} \ell_{-t}^p \ell_{+t}^p - \frac{I_{pt}}{2} \right] + \eta_t^a R\Gamma_t^a \left[ \frac{\sigma_t^2}{2} - \frac{I_{at}^2}{2} \right] \\
+ \eta_t^c R\Gamma_t^c \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t}^c + \frac{\omega_t^2}{2} \ell_{-t}^c \ell_{+t}^c - \frac{I_{ct}^2}{2} \right].
\]

• To avoid exposure to \( \gamma_t \) and \( \omega_t \), consider these option holdings:

\[
\eta_t^p = 0 \quad \eta_t^a = \frac{2}{R\Gamma_t^a} \quad \eta_t^c = 0.
\]

• The position is always long \( \frac{2}{R\Gamma_t^a} \) units of an ATM straddle.

• As spot moves, the no longer ATM straddle must be sold and the freshly minted ATM straddle must be purchased.

• What is the \( \mathbb{Q}_- \) mean gain rate on this high frequency trading strategy in ATM straddles?
Recall that the $\mathbb{Q}_-$ mean gain rate is:

$$G^\gamma_t = \eta^p_t R \Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^p_{-t} + \frac{\omega^2_t}{2} \ell^p_{-t} \ell^p_{+t} - \frac{I^2_{pt}}{2} \right] + \eta^a_t R \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right] + \eta^c_t R \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^c_{-t} + \frac{\omega^2_t}{2} \ell^c_{-t} \ell^c_{+t} - \frac{I^2_{ct}}{2} \right].$$

Also recall the so-called vol trade:

$$\eta^p_t = 0 \quad \eta^a_t = \frac{2}{R \Gamma^a_t} \quad \eta^c_t = 0.$$

Subbing into the top equation implies the following $\mathbb{Q}_-$ mean gain rate:

$$G^\gamma_t \equiv \sigma^2_t - I^2_{at}.$$

If a trader knows that all of the possible realizations of $\sigma_t$ lie above the ATM IV, then the vol trade is profitable on average.
Skew Trade

- We now suppose that $\ell_{c,t} > 0$, $\ell_{p,t} < 0$, and that the put and the call are equally OTM using geometric mean log moneyness:

$$\sqrt{\ell_{t}^p \ell_{t}^p} = \sqrt{\ell_{t}^c \ell_{t}^c} \equiv \ell_{gt}, \quad t \in [0, T].$$

- In this case, the $Q_-$ mean gain rate is:

$$G^v_t = \eta_t^p R\Gamma_t^p \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{t}^p + \frac{\omega_t^2}{2} \ell_{gt}^2 - \frac{l_{pt}^2}{2} \right] + \eta_t^a R\Gamma_t^a \left[ \frac{\sigma_t^2}{2} - \frac{l_{at}^2}{2} \right] + \eta_t^c R\Gamma_t^c \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{t}^c + \frac{\omega_t^2}{2} \ell_{gt}^2 - \frac{l_{ct}^2}{2} \right].$$

- To avoid exposure to $\sigma_t$ and $\omega_t$, consider these option holdings:

$$\eta_t^p = -\frac{1}{(\ell_{c,t} - \ell_{p,t})R\Gamma_t^p}, \quad \eta_t^a = 0, \quad \eta_t^c = \frac{1}{(\ell_{c,t} - \ell_{p,t})R\Gamma_t^c}.$$

viz $\frac{1}{\ell_{c,t} - \ell_{p,t}}$ normalized risk-reversals, each valued at $\frac{C_t(K_t^c)}{R\Gamma_t^c} - \frac{P_t(K_t^p)}{R\Gamma_t^p}$.

- As $S$ moves, both OTM options must be traded.
Skew Trade (Con’d)

- Recall the $\mathbb{Q}_-$ mean gain rate when the put and the call are equally OTM using geometric mean log moneyness:

$$G^\gamma_t = \eta^p_t R\Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^p_t + \frac{\omega^2_t}{2} \ell^2_{gt} - \frac{I^2_{pt}}{2} \right] + \eta^a_t R\Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right]$$

$$+ \eta^c_t R\Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^c_t + \frac{\omega^2_t}{2} \ell^2_{gt} - \frac{I^2_{ct}}{2} \right],$$

and the skew trade: $\eta^p_t = -\frac{1}{(\ell^c_{-t} - \ell^p_{-t})R\Gamma^p_t}, \eta^a_t = 0, \eta^c_t = \frac{1}{(\ell^c_{-t} - \ell^p_{-t})R\Gamma^c_t}$.

- Subbing into the top eq’n implies that this skew trade has the following $\mathbb{Q}_-$ mean gain rate in the value of the position:

$$G^\gamma_t = \gamma_t - \frac{I^2_{ct}}{2} - \frac{I^2_{pt}}{2} \cdot$$

- When all of the possible realizations of $\gamma_t$ lie above the halved implied variance slope, the skew trade is profitable on average.
Smile Trade

• Suppose that the call and put are equally OTM using the \( \ell_- \) measure of log moneyness: \( \ell_{c-t} = -\ell_{p-t} \equiv \ell_{-t} > 0, \ t \in [0, T] \).

• The \( \mathbb{Q}_- \) mean gain rate of the option portfolio simplifies slightly to:

\[
G^\circ_t = \eta^p_t R \Gamma^p_t \left[ \frac{\sigma^2_t}{2} - \gamma_t \ell_{-t} - \frac{\omega^2_t}{2} \ell_{-t} \ell_{+t} - \frac{l^2_{pt}}{2} \right] + \eta^a_t R \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{l^2_{at}}{2} \right]
+ \eta^c_t R \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell_{-t} + \frac{\omega^2_t}{2} \ell_{-t} \ell_{+t} - \frac{l^2_{ct}}{2} \right].
\]

• To avoid exposure to \( \sigma \) and \( \gamma \), consider these option holdings:

\[
\eta^p_t = \frac{1}{\bar{\ell}_{agt}^2} R \Gamma^p_t \quad \eta^a_t = -\frac{2}{\bar{\ell}_{agt}^2} R \Gamma^a_t \quad \eta^c_t = \frac{1}{\bar{\ell}_{agt}^2} R \Gamma^c_t \quad t \in [0, T],
\]

where \( \bar{\ell}_{agt} \equiv \sqrt{\frac{\ell_{c-t} + |\ell_{-t}|}{2} \frac{\ell_{c-t} + |\ell_{+t}|}{2}} \).

• This is a long position of \( \frac{1}{\bar{\ell}_{agt}^2} \) normalised butterflies, each with value

\[
\frac{C_t(K^c_t)}{R \Gamma^c_t} - 2 \frac{A_t(K^p_t)}{R \Gamma^a_t} + \frac{P_t(K^p_t)}{R \Gamma^p_t}.
\]
Smile Trade (Con’d)

• Recall the $\mathbb{Q}_-$ mean gain rate of the option portfolio when the call and put are equally OTM using the $\ell_-$ measure of log moneyness:

$$G^\gamma_t = \eta^p_t R \Gamma^p_t \left[ \frac{\sigma^2_t}{2} - \frac{\omega^2_t}{2} - \frac{I^2_{pt}}{2} \right] + \eta^a_t R \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right] + \eta^c_t R \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \frac{\omega^2_t}{2} - \frac{I^2_{ct}}{2} \right].$$

• Also recall the smile trade:

$$\eta^p_t = \frac{1}{\ell_{agt}^2 R \Gamma^p_t}, \quad \eta^a_t = -\frac{2}{\ell_{agt}^2 R \Gamma^a_t}, \quad \eta^c_t = \frac{1}{\ell_{agt}^2 R \Gamma^c_t} \quad t \in [0, T].$$

• Substituting into the top equation implies that this smile trade has

$$G^\gamma_t = \omega^2_t - \frac{I^2_{ct} + I^2_{pt}}{2 \ell_{agt}^2} - \frac{I^2_{at}}{2}.$$

• When all possible realizations of $\omega^2_t$ lie above the convexity measure of halved implied variance, the smile trade is profitable on average.
Summary

• We assumed \( dS_t = \sigma_t S_t dW_t, \ dl_t(K) = \omega_t l_t(K) dZ_t \), for \( K > 0, t \in [0, T] \), where \( W, Z \) are \( \mathbb{Q} \)-SBM's with unknown random correlation \( \rho_t \in [-1, 1] \).

• By dynamically trading ATM straddles, normalized risk reversals, and normalized butterfly spreads, a trader was able to synthesize a short-term forward contract on \( \sigma_t^2 \), on \( \gamma_t \equiv \sigma_t \rho_t \omega_t \), and on \( \omega_t^2 \).

• The forward price paid for each of these three bets was either \( l_{at}^2 \) (vol trade), \( \frac{l_{ct}^2 - l_{pt}^2}{\ell_{c_t}^2 - \ell_{p_t}^2} \) (skew trade), or \( \frac{l_{ct}^2 + l_{pt}^2}{\ell_{ag}^2} - l_{at}^2 \) (smile trade).

• The three forward prices measure level, slope, and convexity of a (halved) implied variance curve.

• The paper shows that the three standard FX option quotes are positive multiples of the vega of normalized ATM straddles, normalized risk-reversals, and normalized butterfly-spreads.

• Thanks for listening!